Adaptive Conformal Predictions for Time Series

An application to forecasting French electricity Spot prices

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Forecasting French electricity Spot prices

Electricity Spot prices

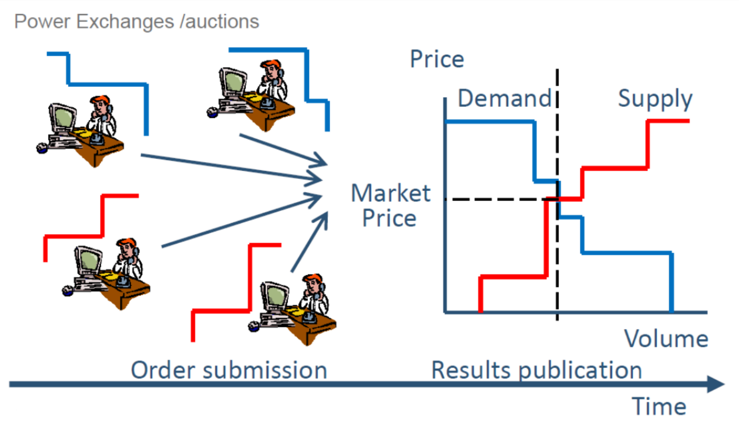

Figure 1: Drawing of spot auctions mechanism

French Electricity Spot prices data set: visualisation

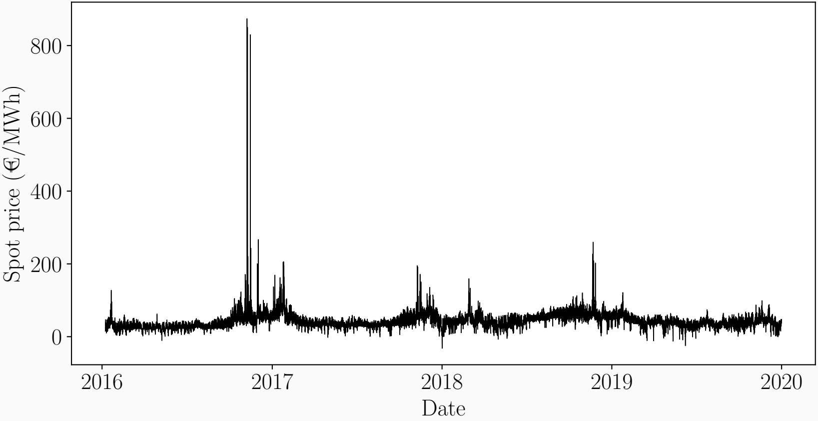

Figure 2: Representation of the French electricity spot price, from 2016 to 2019.

French Electricity Spot prices data set: extract

Date and time	Price	Price D-1	Price D-7	For. cons.	DOW
11/01/16 0PM	21.95	15.58	13.78	58800	Monday
11/01/16 1PM	20.04	19.05	13.44	57600	Monday
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12/01/16 1PM	19.81	20.04	24.42	59800	Tuesday
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18/01/16 0PM	38.14	37.86	21.95	70400	Monday
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Forecasting French electricity Spot prices

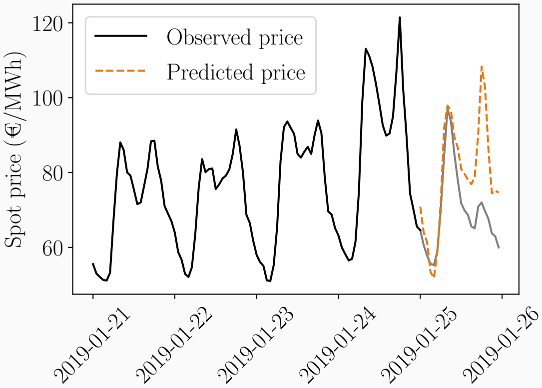


Figure 3: French electricity spot price and its prediction with random forest.

↪ $(x_t, y_t) \in \mathbb{R}^d \times \mathbb{R}$ ($d = 56$, details later)

↪ 3 years for training

↪ 1 year to forecast

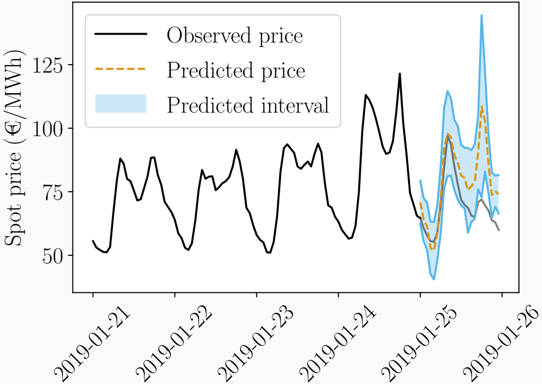


Figure 4: French electricity spot price, its prediction and its uncertainty with AgACI (proposed algorithm).

- Target coverage: 90%
- Empirical coverage: 91.68%

Conformal prediction and time series, what's the issue?

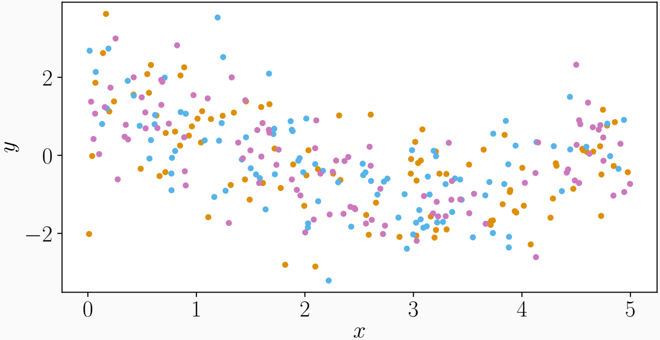
Framework and notations

- Data: T_0 observations $(x_1, y_1), \ldots, (x_{T_0}, y_{T_0})$ in $\mathbb{R}^d \times \mathbb{R}$
- Aim: predict the response values as well as predictive intervals for T_1 subsequent observations $x_{T_0+1}, \ldots, x_{T_0+T_1}$

$\hookrightarrow$ Build the smallest interval $\mathcal{C}_\alpha^t$ such that:

$$\mathbb{P}\left\{ Y_t \in \mathcal{C}_\alpha^t(X_t) \right\} \geq 1 - \alpha, \text{ for } t \in [\![T_0 + 1, T_0 + T_1]\!].$$

Split Conformal Prediction (Vovk et al., 2005): scheme



Randomly split the data to obtain a proper training set and a calibration set. Keep the test set.

Split Conformal Prediction (Vovk et al., 2005): scheme

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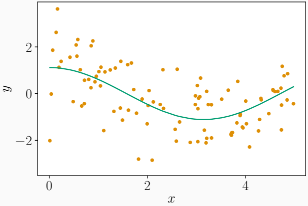

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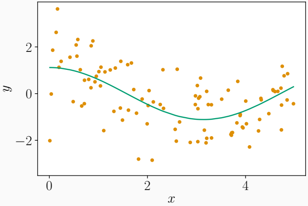

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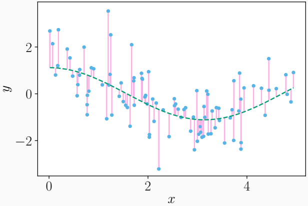

- Predict with $\hat{\mu}$.
- Get the residuals $\hat{\varepsilon}_i$ and form the scores $s_i = |\hat{\varepsilon}_i|$.
- Get their $(1-\alpha) \times (1 + \frac{1}{\#\text{Cal}})$ empirical quantile: $Q_{1-\hat{\alpha}}(s_i)$.

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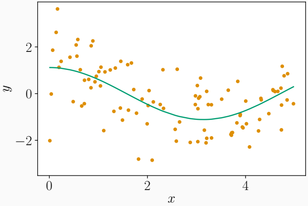
1) 

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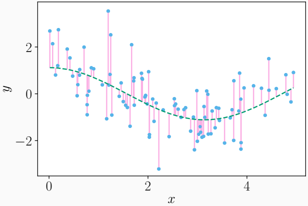
2) 

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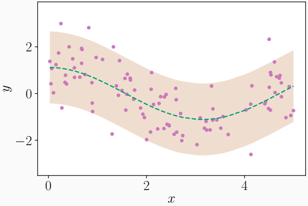
3) 

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- Build $\hat{C}_{\hat{\alpha}}(x)$: $[\hat{\mu}(x) \pm Q_{1-\hat{\alpha}}(s_i)]$.

Split Conformal Prediction: guarantees

Split conformal prediction is simple to compute and satisfies:

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- distribution-free as long as the data is exchangeable;
  - $\hookrightarrow$ the scores need to be exchangeable (but then it would not work with any regression algorithm)
- in finite sample.

Time series are not exchangeable

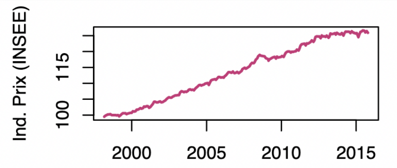

Figure 5: Trend[1]

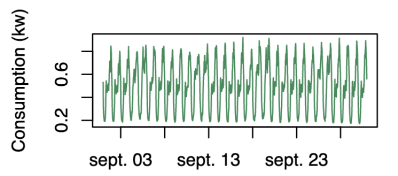

Figure 6: Seasonality[1]

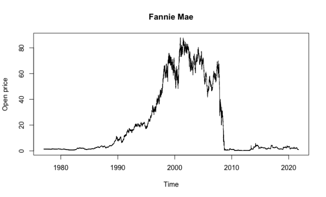

Figure 7: Shift

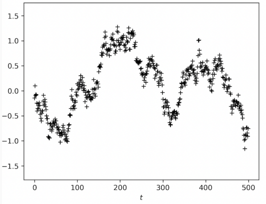

Figure 8: Time dependence

[1] Images from Yannig Goude class material.

Non-exchangeable even if the noise is exchangeable

Assume the following model:

$$Y_t = f_t(X_t) + \varepsilon_t, \text{ for } t \in \mathbb{N}^*,$$

for some function f_t, and some noise ε_t.

Even if the noise ε_t is exchangeable, we can produce dependent residuals.

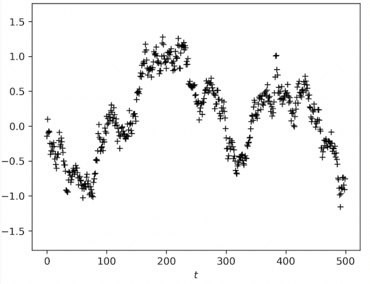

Figure 9: Auto-Regressive residuals

Available methods for non-exchangeable data, in the context of time series

How to adapt to time series?

Usual ideas from the time series literature:

- Consider an online procedure (for each new data, re-train and re-calibrate)

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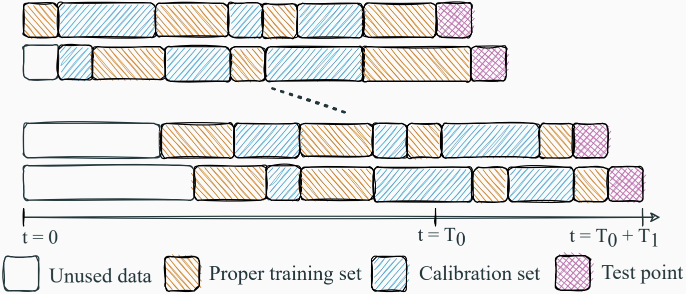

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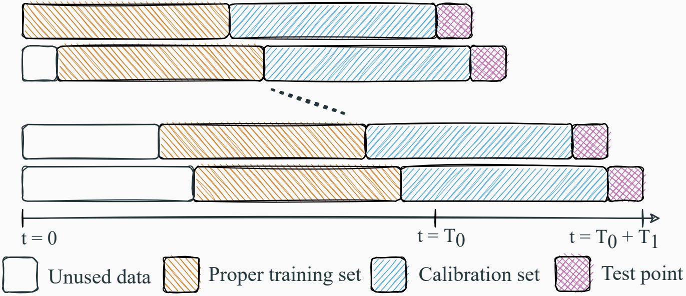

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- Consider an online procedure (for each new data, re-train and re-calibrate)
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- Use a sequential split
  - ↪ use only the past so as to correctly estimate the variance of the residuals (using the future leads to optimistic residuals and underestimation of their variance)

Online sequential split conformal prediction (OSSCP)

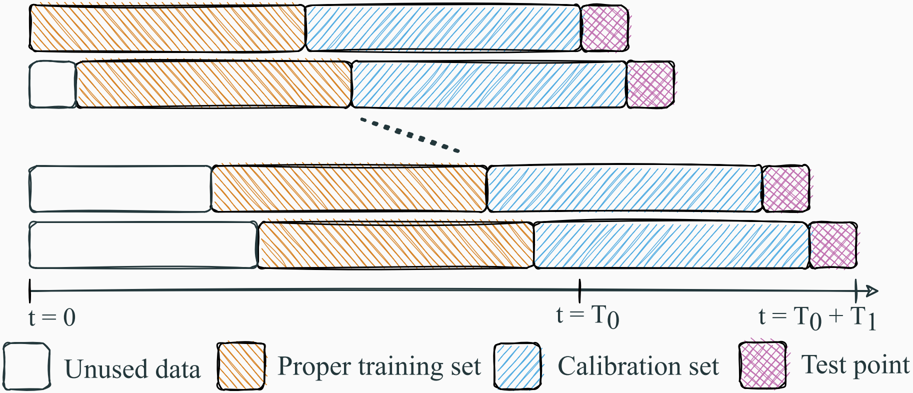
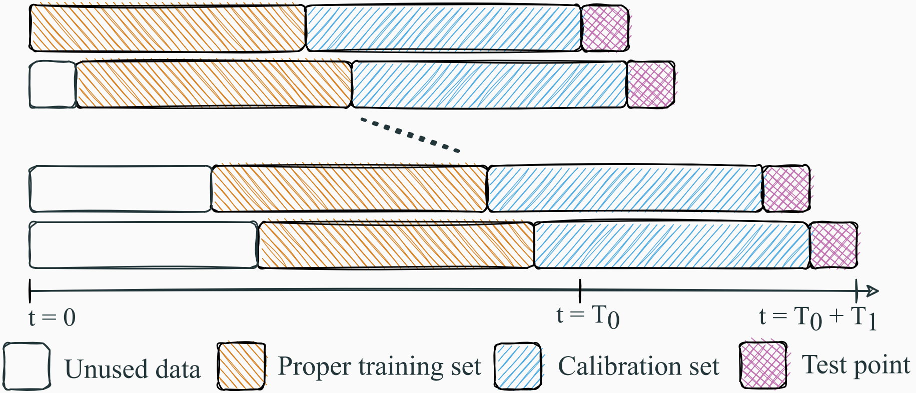


Figure 10: Diagram describing the online sequential split conformal prediction.



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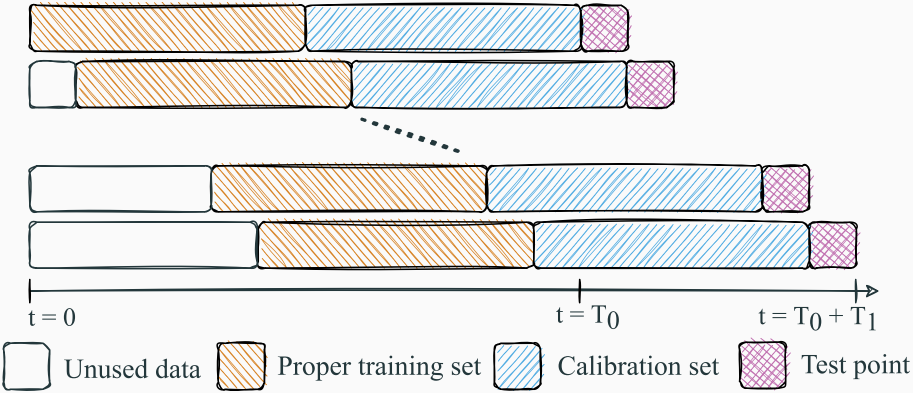

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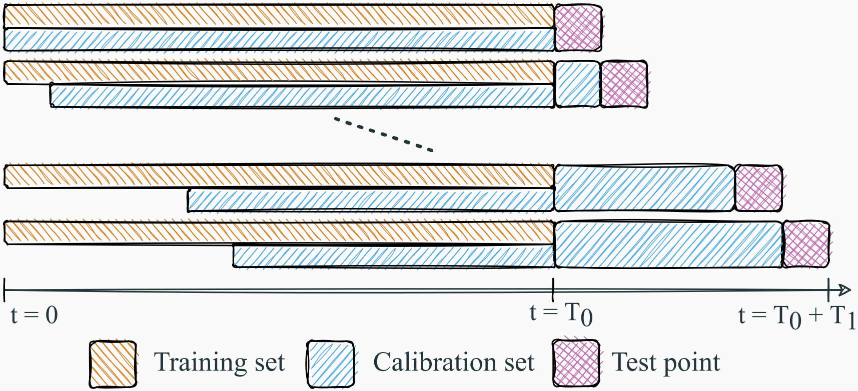

Figure 11: Diagram describing the EnbPI algorithm.

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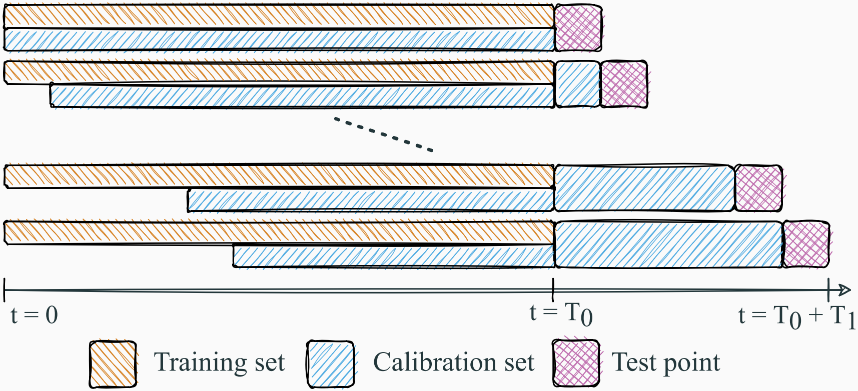

Figure 11: Diagram describing the EnbPI algorithm.

↪ tested on other real time series

↪ compared to offline methods

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The proposed update scheme is the following:

$$\alpha_{t+1} := \alpha_t + \gamma\left(\alpha - \text{error}_t\right) \tag{1}$$

with:

$$\text{error}_t := \begin{cases} 1 \text{ if } y_t \notin \hat{\mathcal{C}}_{\alpha_t}\left(x_t\right), \\ 0 \text{ otherwise }, \end{cases}$$

and $\alpha_1 = \alpha$, $\gamma \geq 0$.

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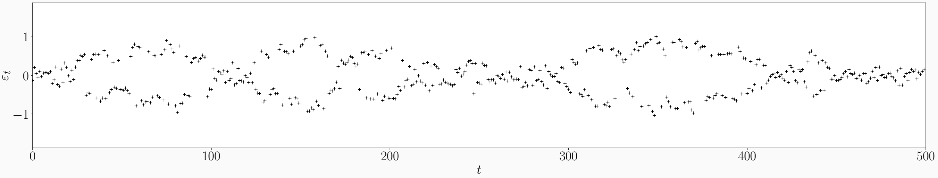
Intuition: if we did make an error, the interval was too small so we want to increase its length by taking a higher quantile (a smaller α_t). Reversely if we included the point.

Visualisation of the procedure



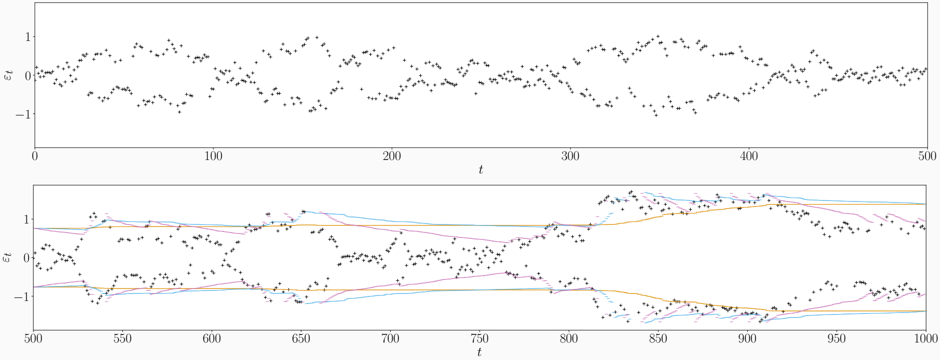


Figure 12: Visualisation of ACI with different values of γ ($\gamma = 0$, $\gamma = 0.01$, $\gamma = 0.05$)

ACI asymptotic result

Gibbs and Candès (2021) provide an asymptotic validity result for any sequence of observations.

$$\frac{1}{T_1}\sum_{t=T_0+1}^{T_0+T_1} \mathbb{1}\left\{y_t \in \hat{\mathcal{C}}_{\alpha_t}\left(x_t\right)\right\} \underset{T_1\to+\infty}{\longrightarrow} 1-\alpha \quad \textit{e.g. 90\%}$$

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$\Rightarrow$ favors large γ. But, the higher γ, the more frequent are the infinite intervals.

Theoretical analysis of ACI's length

Approach

Aim: derive theoretical results on the **average length** of ACI depending on γ

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$\hookrightarrow$ Guideline for choosing γ

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- consider extreme cases (useful in an online context) with simple theoretical distributions
  1. exchangeable
  2. Auto-Regressive case (AR(1))
- Assume the calibration is perfect (and more), to rely on Markov Chain theory

Theoretical analysis of ACI's length: exchangeable case

Define $L(\alpha_t) = 2Q(1-\alpha_t)$ the length of the interval predicted by the adaptive algorithm at time t, and $L_0 = 2Q(1-\alpha)$ the length of the interval predicted by the non-adaptive algorithm ($\gamma = 0$).

Theorem

Assume the scores are exchangeable with quantile function Q perfectly estimated at each time, and other assumptions.

Then, for all $\gamma > 0$, $(\alpha_t)_{t>0}$ forms a Markov Chain, that admits a stationary distribution π_γ, and

$$\frac{1}{T}\sum_{t=1}^{T} L(\alpha_t) \xrightarrow[T\to+\infty]{a.s.} \mathbb{E}_{\pi_\gamma}[L] \overset{not.}{=} \mathbb{E}_{\tilde{\alpha}\sim\pi_\gamma}[L(\tilde{\alpha})].$$

Moreover, as $\gamma \to 0$,

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Theoretical analysis of ACI's length: exchangeable case

Theorem (Informal)

If the residuals are exchangeable and if the calibration is perfect, then as $\gamma \to 0$:

$$\text{Average length of intervals from ACI using } \gamma$$
$$=$$
$$\text{Average length of intervals from Split Conformal Prediction}$$
$$+ \gamma \times \mathcal{C}(\alpha, \text{distribution of the data}),$$

where $\mathcal{C}(\alpha, \text{distribution of the data}) > 0$ in non-atypical cases.

Numerical analysis of ACI's length: AR(1) case

Theorem

Assume the residuals follow an AR(1) process: $\hat{\varepsilon}_{t+1} = \varphi \hat{\varepsilon}_t + \xi_{t+1}$ *with* $(\xi_t)_t$ *i.i.d. random variables and other assumptions, we have:*

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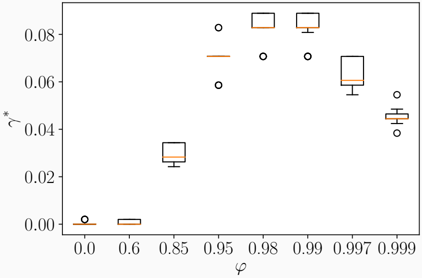

Figure 13: γ^* minimizing the average length for each φ.

AgACI

AgACI: adaptive wrapper around ACI

Online aggregation under expert advice (Cesa-Bianchi and Lugosi, 2006) computes an optimal weighted mean of experts.

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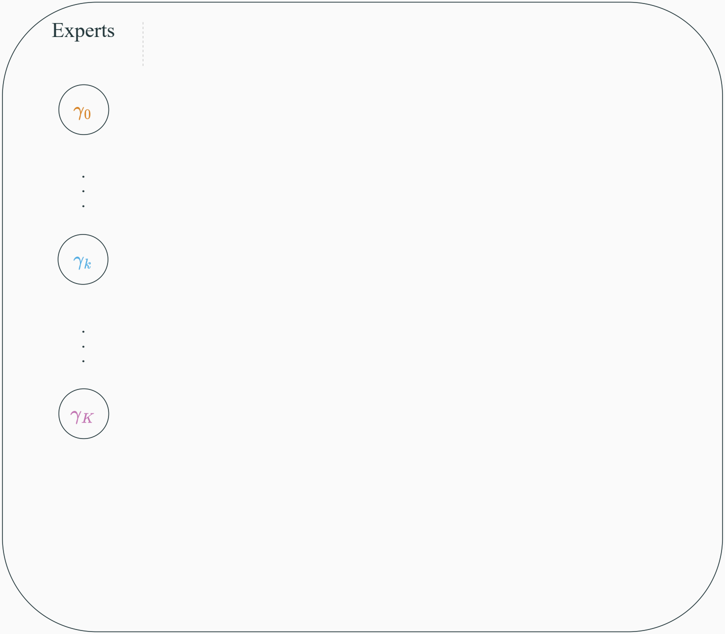
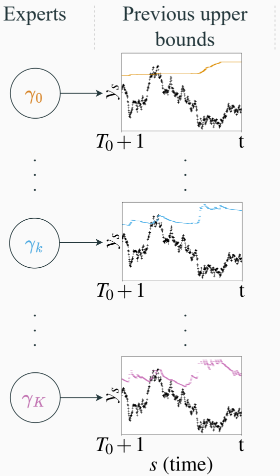
`AgACI` performs 2 independent aggregations: one for each bound (the upper and lower ones).


Experts
γ_0
γ_k
γ_K

AgACI: adaptive wrapper around ACI, scheme (upper bound)



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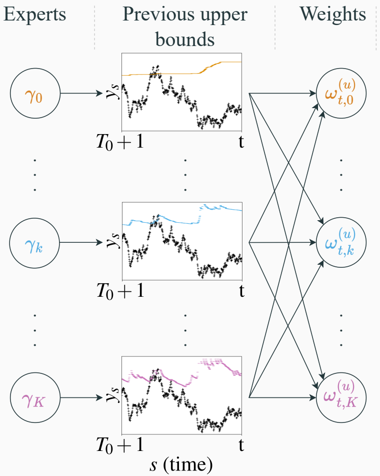

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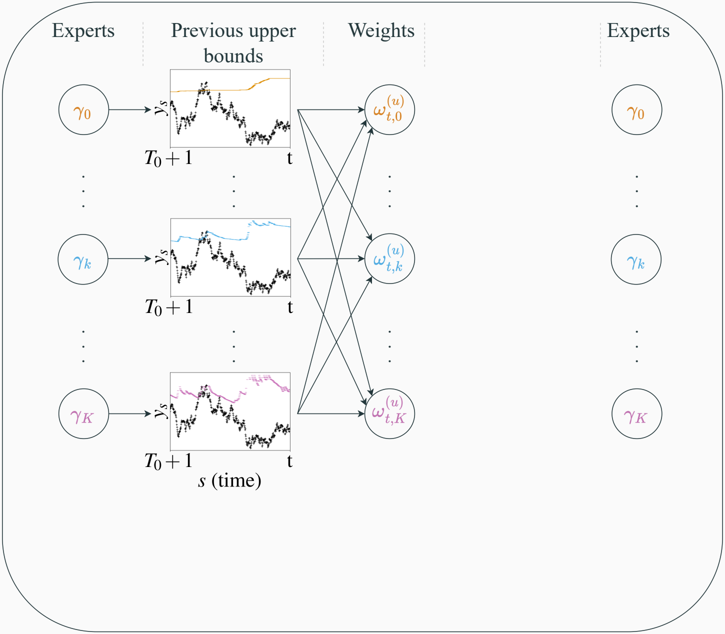

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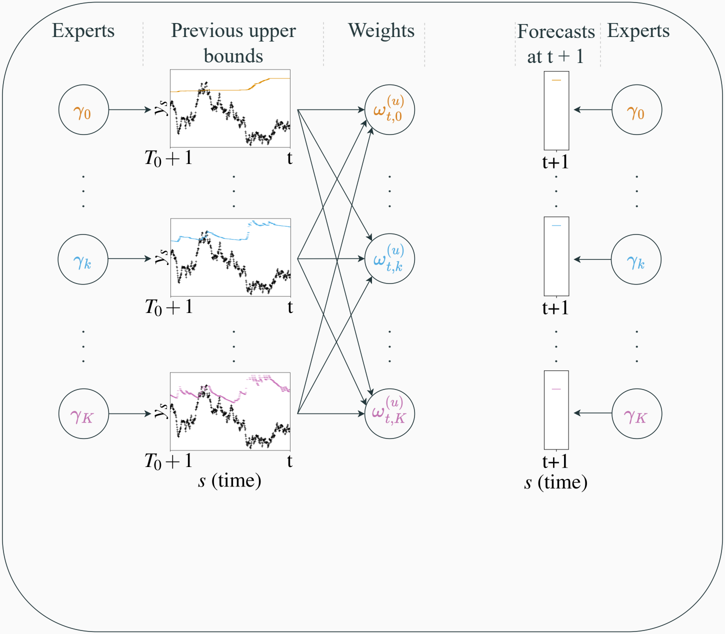

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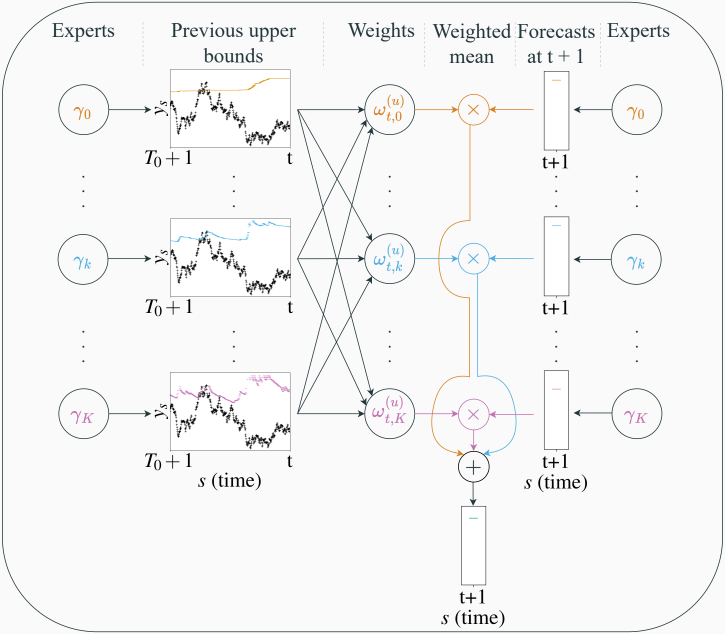

Comparison on simulated data

Data generation and simulation settings

$$Y_t = 10\sin\left(\pi X_{t,1} X_{t,2}\right) + 20\left(X_{t,3} - 0.5\right)^2 + 10X_{t,4} + 5X_{t,5} + \varepsilon_t$$

Data generation and simulation settings

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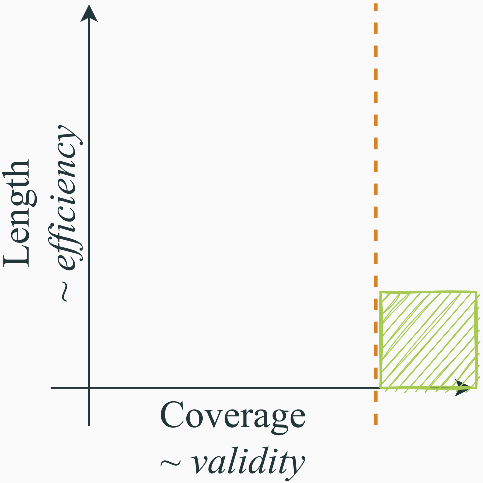
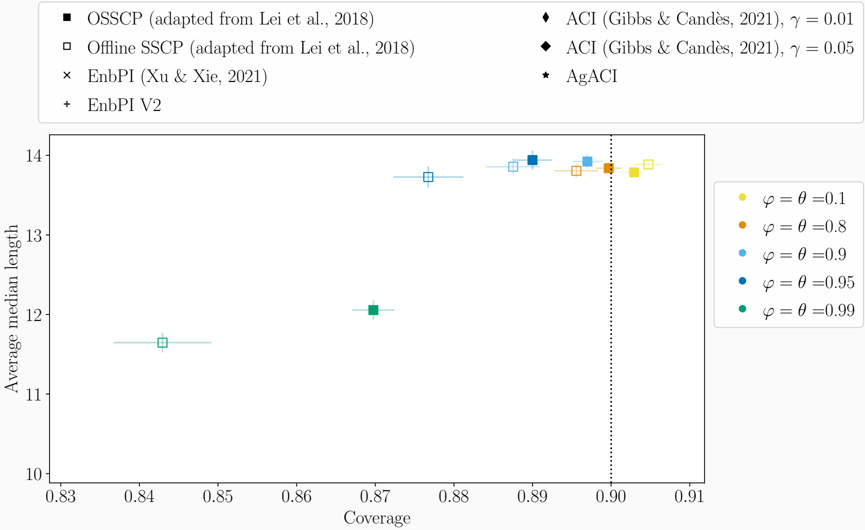
- We fix σ to keep the variance Var(ε_t) constant to 10 (or 1).
- We use random forest as regressor.
- For each setting (pair variance and φ,θ):
  - 300 points, the last 100 kept for prediction and evaluation,
  - 500 repetitions,

  $\Rightarrow$ in total, $100 \times 500 = 50000$ predictions are evaluated.

Visualisation of the results



Results: impact of the temporal dependence, ARMA(1,1), variance 10



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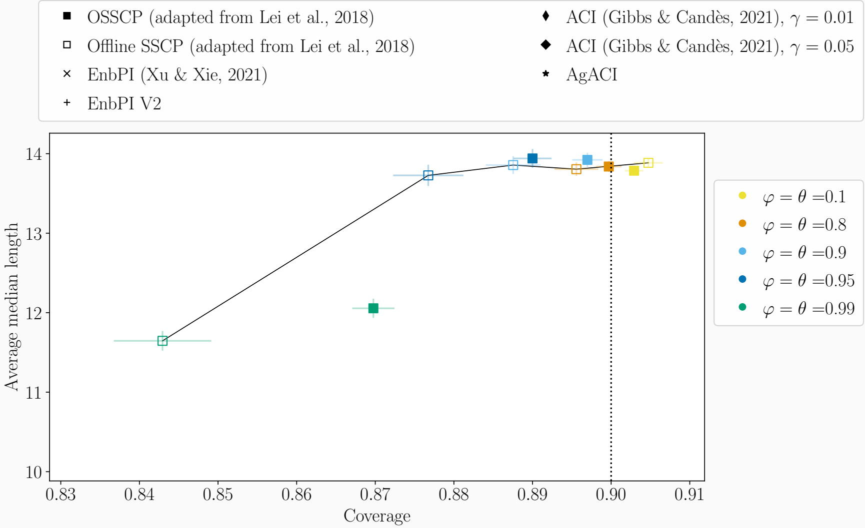

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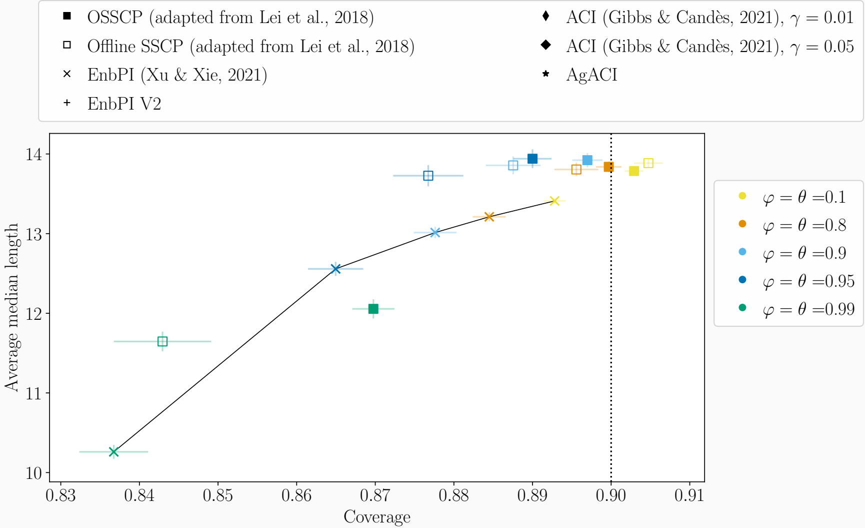

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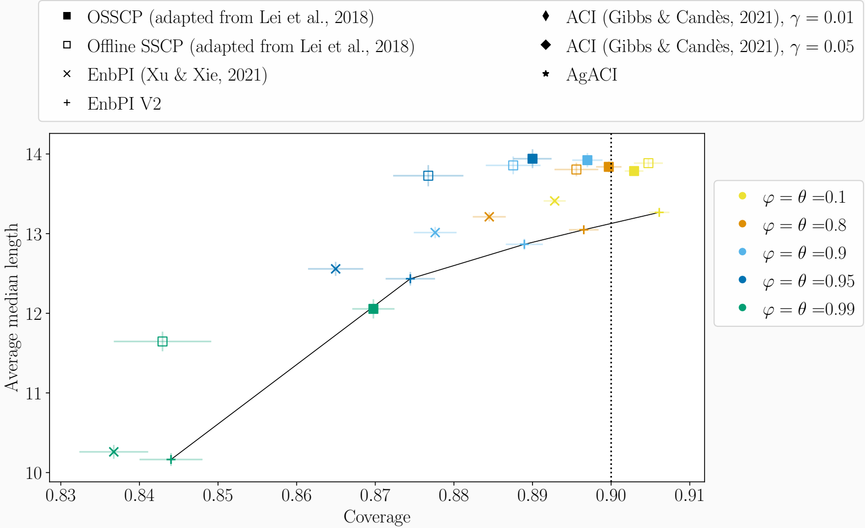

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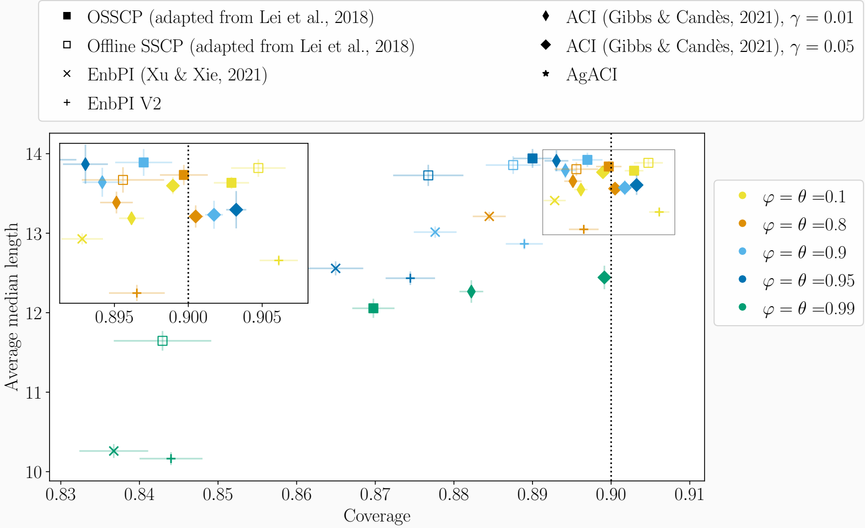

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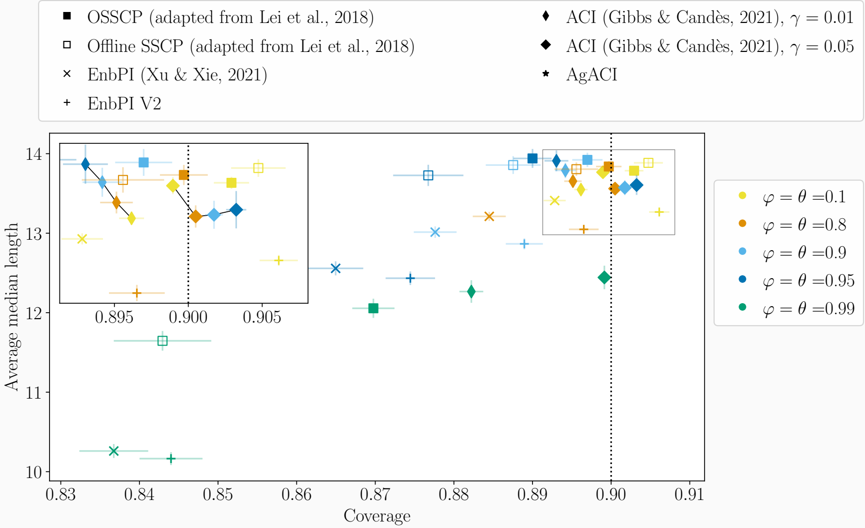

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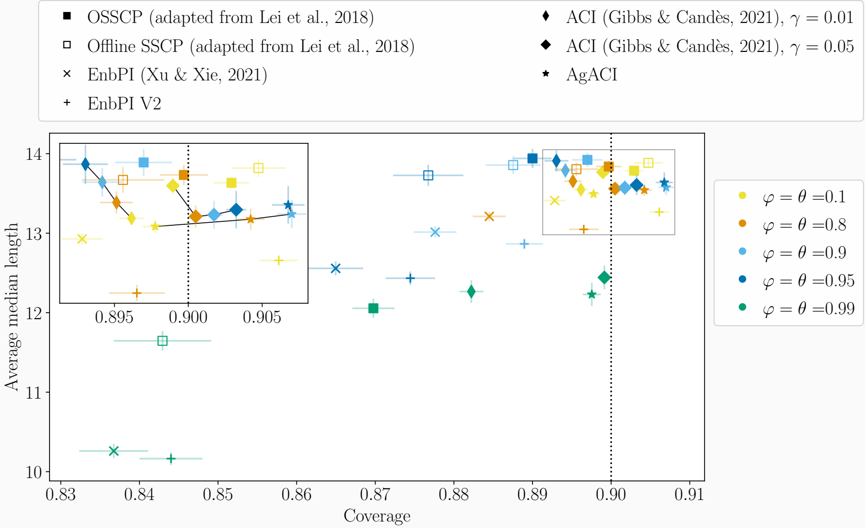

Summary

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6. **AgACI.** Achieves *valid* coverage for every simulation settings, with good *efficiency*.

Price prediction with confidence in 2019

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- Forecast for the year 2019.
- Random forest regressor.
- One model per hour, we concatenate the predictions afterwards.

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Forecasted consumption

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Forecasted consumption

Encoded day of the week

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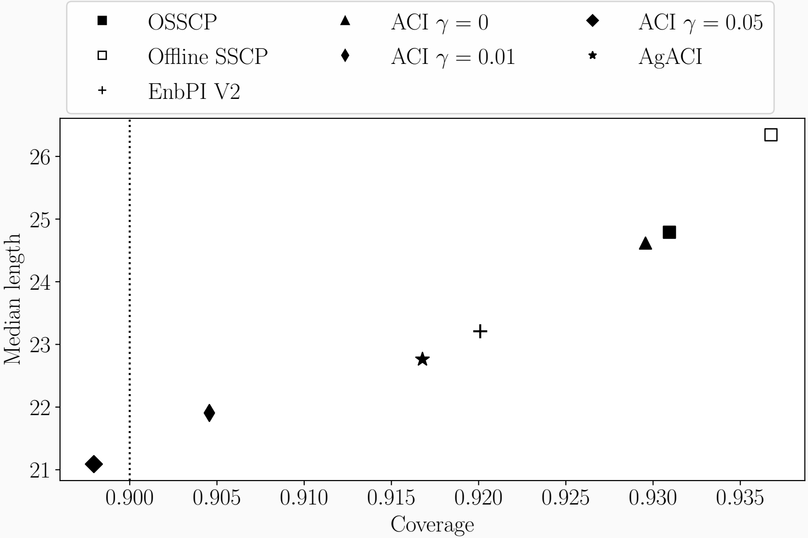
  - 3 years for training/calibration, i.e. $T_0 = 1096$ observations
  - 1 year to forecast, i.e. $T_1 = 365$ observations

Performance on predicted French electricity Spot price for the year 2019



Performance on predicted French electricity Spot price: visualisation of a day

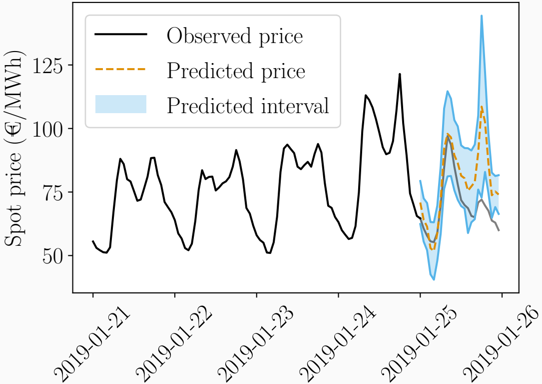


Figure 14: French electricity spot price, its prediction and its uncertainty with AgACI.

Be careful at conditional guarantees!

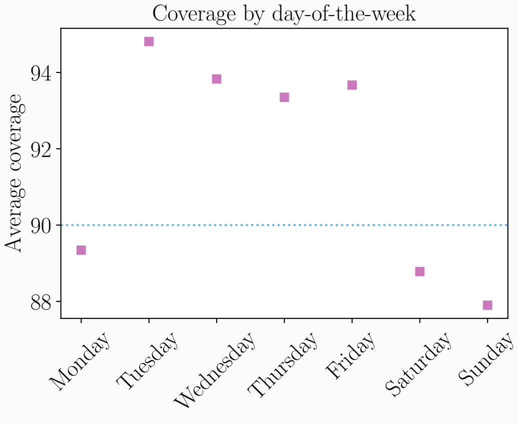


Figure 15: Empirical coverage of AgACI depending on the day-of-the-week.

Concluding remarks

Contributions and messages

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- ACI useful for time series with general dependency (extensive synthetic experiments and real data)
- Empirical proposition of an adaptive choice of γ: `AgACI`

Thank you! Questions?

Cesa-Bianchi, N. and Lugosi, G. (2006). *Prediction, learning, and games*. Cambridge University Press.

Gibbs, I. and Candès, E. (2021). Adaptive Conformal Inference Under Distribution Shift. *arXiv:2106.00170 [stat]*. arXiv: 2106.00170.

Kath, C. and Ziel, F. (2021). Conformal prediction interval estimation and applications to day-ahead and intraday power markets. *International Journal of Forecasting*, 37(2):777–799.

Vovk, V., Gammerman, A., and Shafer, G. (2005). *Algorithmic Learning in a Random World*. Springer US.

Wisniewski, W., Lindsay, D., and Lindsay, S. (2020). Application of conformal prediction interval estimations to market makers' net positions. In Gammerman, A., Vovk, V., Luo, Z., Smirnov, E., and Cherubin, G., editors, *Proceedings of the Ninth Symposium on Conformal and Probabilistic Prediction and Applications*, volume 128 of *Proceedings of Machine Learning Research*, pages 285–301. PMLR.

Xu, C. and Xie, Y. (2021). Conformal prediction interval for dynamic time-series. In Meila, M. and Zhang, T., editors, *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pages 11559–11569. PMLR.

Examples of non-exchangeable scores with exchangeable noise

Endogenous and not perfectly estimated

Assume $X_t = Y_{t-1} \in \mathbb{R}$ and that

$$Y_t = aY_{t-1} + \varepsilon_t,$$

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Then, for any t, we have that:

$$\hat{\varepsilon}_t = Y_t - \hat{Y}_t = (a - \hat{a})\, Y_{t-1} + \varepsilon_t$$

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$\hat{\varepsilon}_t$ is an ARMA process of parameters $\varphi = a$ and $\theta = -\hat{a}$.

Thus, we have generated dependent residuals (ARMA residuals) even if the underlying model only had white noise.

Exogenous and misspecified

Assume $X_t \in \mathbb{R}^2$ and that:

$$Y_t = aX_{1,t} + bX_{2,t} + \varepsilon_t,$$

with $\varepsilon_t \underset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, $X_{2,t+1} = \varphi X_{2,t} + \xi_t$, $\xi_t \underset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $X_{1,t}$ can be any random variable.

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Thus, we have generated dependent residuals (auto-regressive residuals) even if the underlying model only had i.i.d. Gaussian noise.

Analysis of ACI's efficiency depending on γ

Numerical analysis of ACI's length: AR(1) case

Assume the residuals follow an AR(1) process: $\hat{\varepsilon}_{t+1} = \varphi\hat{\varepsilon}_t + \xi_{t+1}$ with $(\xi_t)_t$ i.i.d. random variables and other assumptions, we have:

$$\frac{1}{T}\sum_{t=1}^{T} L(\alpha_t) \xrightarrow[T\to+\infty]{a.s.} \mathbb{E}_{\pi_{\gamma,\varphi}}[L].$$

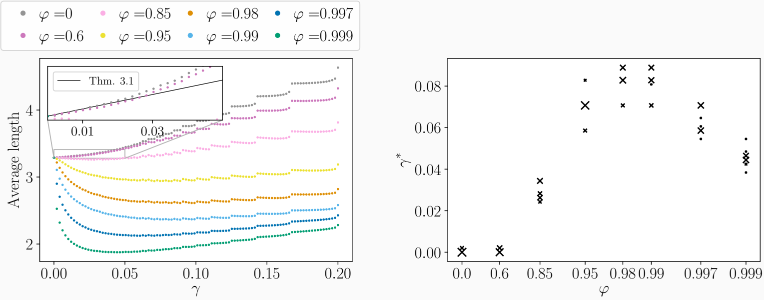


Figure 16: Left: evolution of the mean length depending on γ for various φ. Right: γ^* minimizing the average length for each φ.

EnbPI

EnbPI, Xu and Xie (2021)

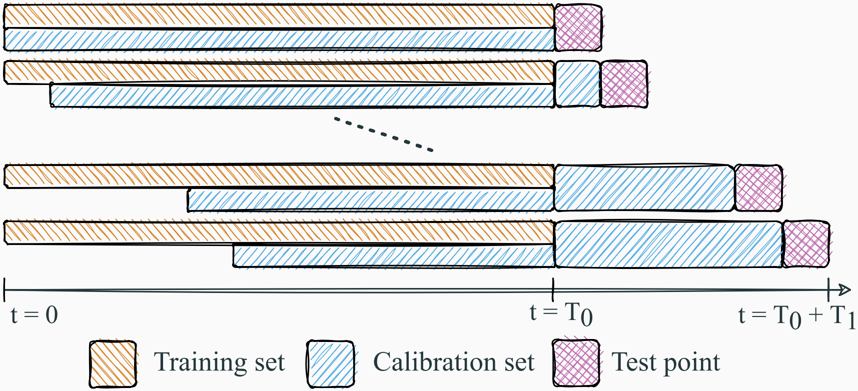
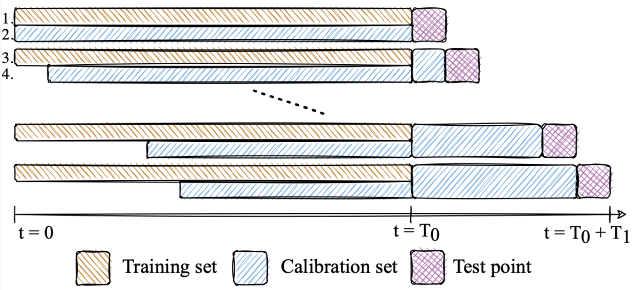


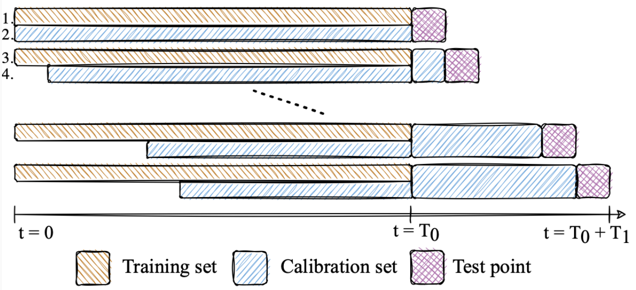
Figure 17: Diagram describing the EnbPI algorithm.

EnbPI, Xu and Xie (2021)



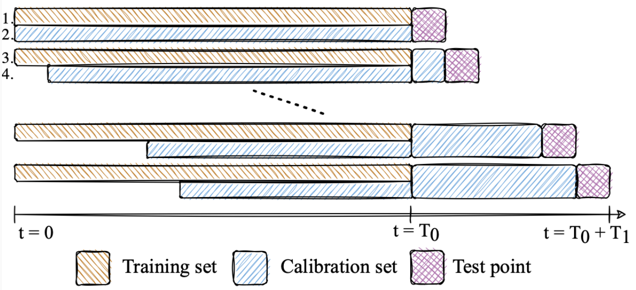
1. Train B bootstrap predictors;

EnbPI, Xu and Xie (2021)



1. Train B bootstrap predictors;
2. Obtain out-of-bootstrap residuals by aggregating the corresponding predictors;

EnbPI, Xu and Xie (2021)



1. Train B bootstrap predictors;
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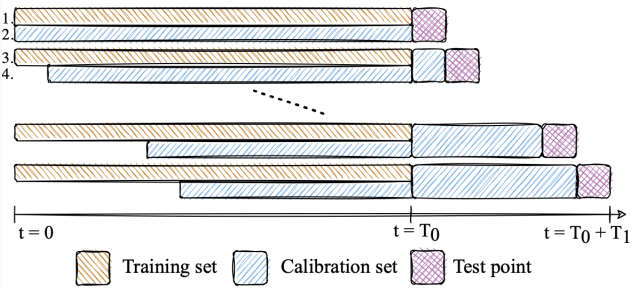
3. Do not re-train the B bootstrap predictors;

EnbPI, Xu and Xie (2021)



1. Train B bootstrap predictors;
2. Obtain out-of-bootstrap residuals by aggregating the corresponding predictors;
3. Do not re-train the B bootstrap predictors;
4. Obtain new residual by aggregating all the predictors. Forget the first residuals.

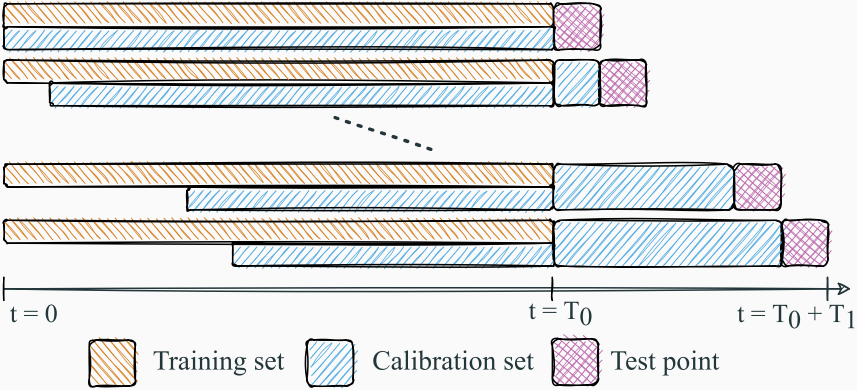


Figure 17: Diagram describing the EnbPI algorithm.

EnbPI (ICML, Xu and Xie, 2021) aggregates with 2 different functions.

EnbPI, Xu and Xie (2021)

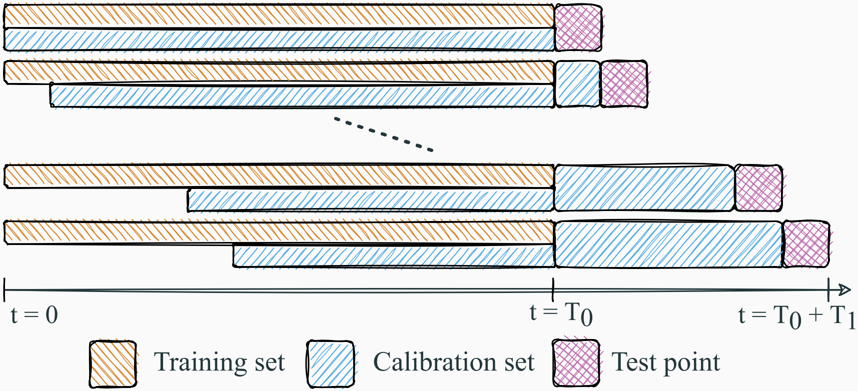


Figure 17: Diagram describing the EnbPI algorithm.

EnbPI (ICML, Xu and Xie, 2021) aggregates with 2 different functions.

⇒ We propose `EnbPI V2` with the same aggregation function all along (similar to EnbPI on last ArXiV version from Xu and Xie).

EnbPI, Xu and Xie (2021)

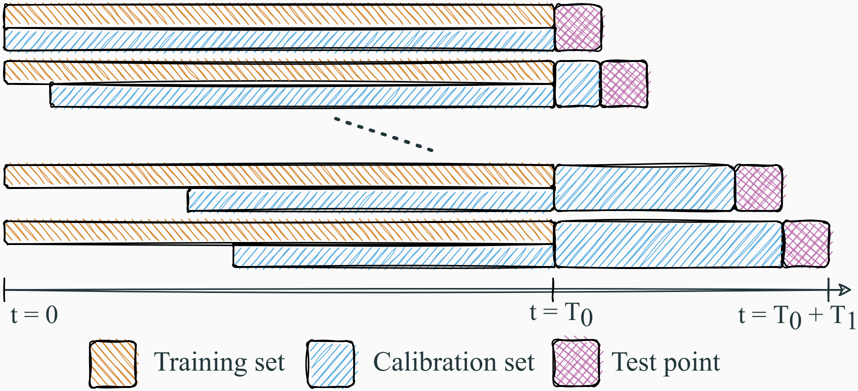


Figure 17: Diagram describing the EnbPI algorithm.

↪ tested on other real time series

↪ compared to offline methods

Details on the simulation set up

Data generation

$$Y_t = 10\sin(\pi X_{t,1} X_{t,2}) + 20(X_{t,3} - 0.5)^2 + 10X_{t,4} + 5X_{t,5} + \varepsilon_t$$

where the X_t are multivariate uniformly distributed on $[0, 1]$ and ε_t are generated from an ARMA(1,1) process.

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where the X_t are multivariate uniformly distributed on $[0, 1]$ and ε_t are generated from an ARMA(1,1) process.

$\Rightarrow$ dependence structure in the noise in order to:

- control the strength of the scores dependence,
- evaluate the impact of this temporal dependence structure of the results.

Auto-Regressive Moving Average

Definition (ARMA(1,1) process)

We say that ε_t is an ARMA(1,1) process if for any t:

$$\varepsilon_{t+1} = \varphi\varepsilon_t + \xi_{t+1} + \theta\xi_t,$$

with:

- $\theta + \varphi \neq 0$, $|\varphi| < 1$ and $|\theta| < 1$;
- ξ_t is a white noise of variance σ^2, called the **innovation**.

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- The higher φ and θ, the stronger the dependence.
- The asymptotic variance of this process is:

$$\mathsf{Var}(\varepsilon_t) = \sigma^2 \frac{1 - 2\varphi\theta + \theta^2}{1 - \varphi^2}.$$

- If $\theta = 0$, only the auto-regressive part, it is an AR(1).
- If $\varphi = 0$, only the moving-average part, it is an MA(1).

Simulation settings

- φ and θ range in $[0.1, 0.8, 0.9, 0.95, 0.99]$.
- We fix σ so as to keep the variance $\mathrm{Var}(\varepsilon_t)$ constant to 1 or 10.
- We use random forest as regressor.

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For each setting:

- 300 points, the last 100 kept for prediction and evaluation,
- 500 repetitions,

$\Rightarrow$ in total, $100 \times 500 = 50000$ predictions are evaluated.

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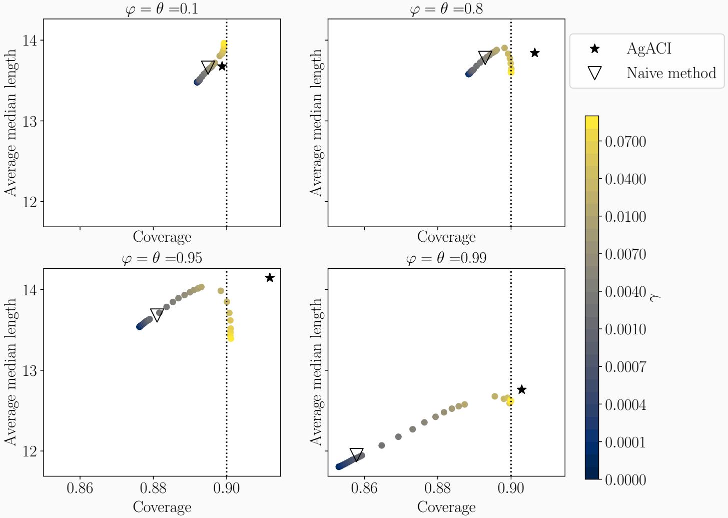
We present the results in the ARMA(1,1) case, but we also have them for AR(1) and MA(1) processes.

Additional results on the synthetic data sets

Empirical evaluation of ACI sensitivity to γ and adaptive choice



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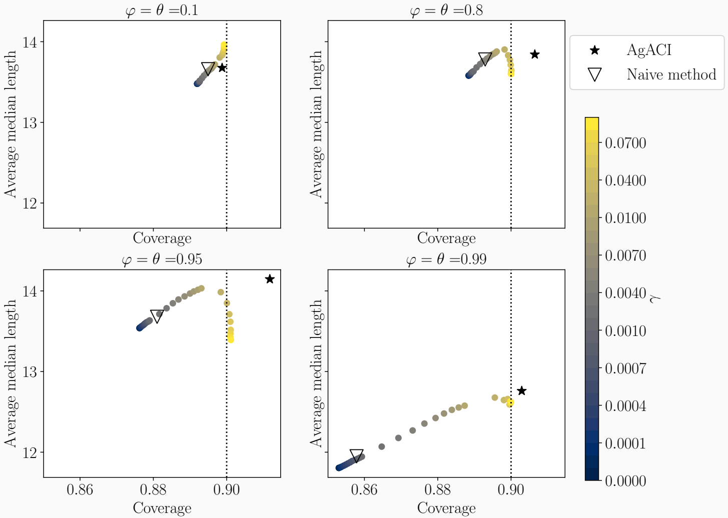
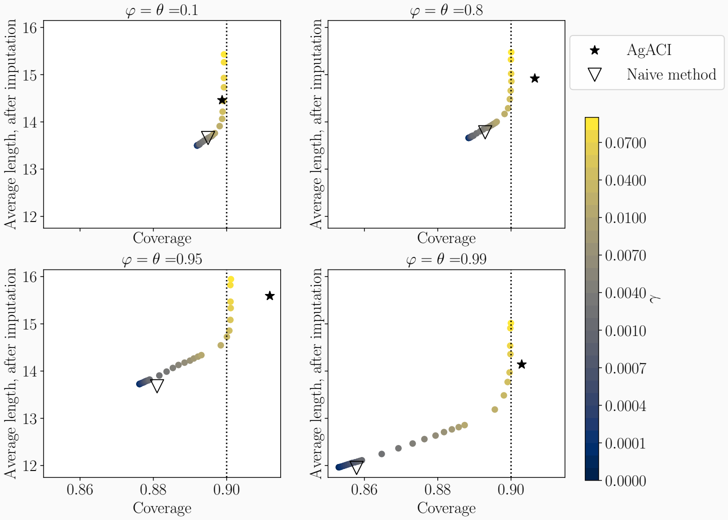


$\Rightarrow$ The more the dependence, the more sensitive to γ is ACI.

Empirical evaluation of ACI sensitivity to γ and adaptive choice



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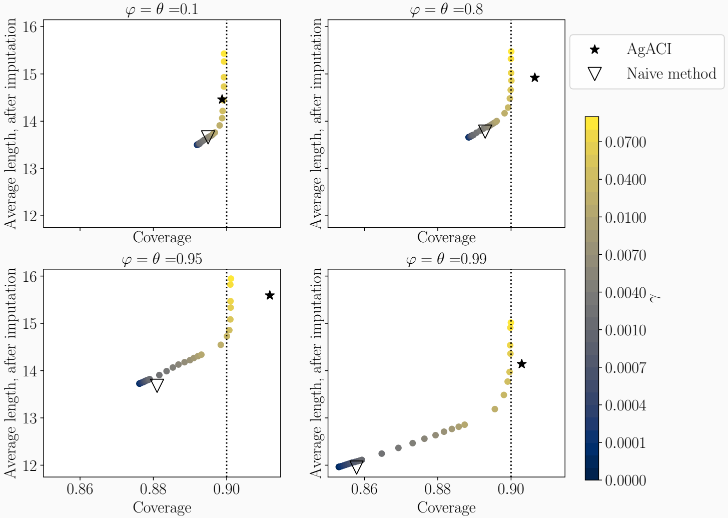
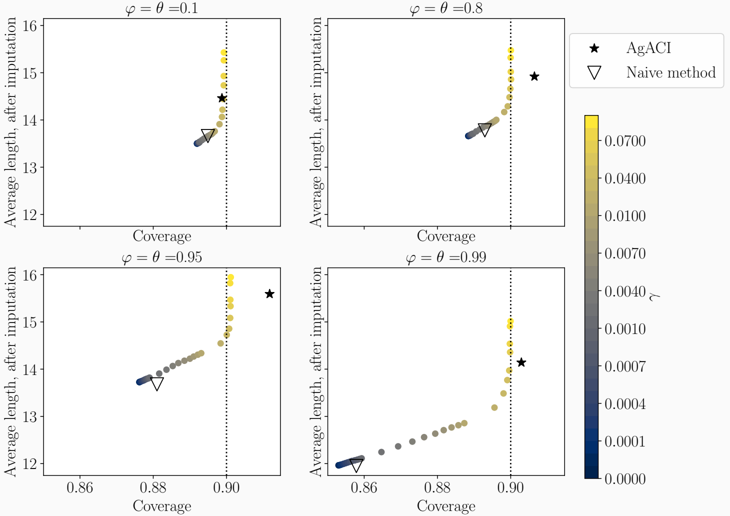


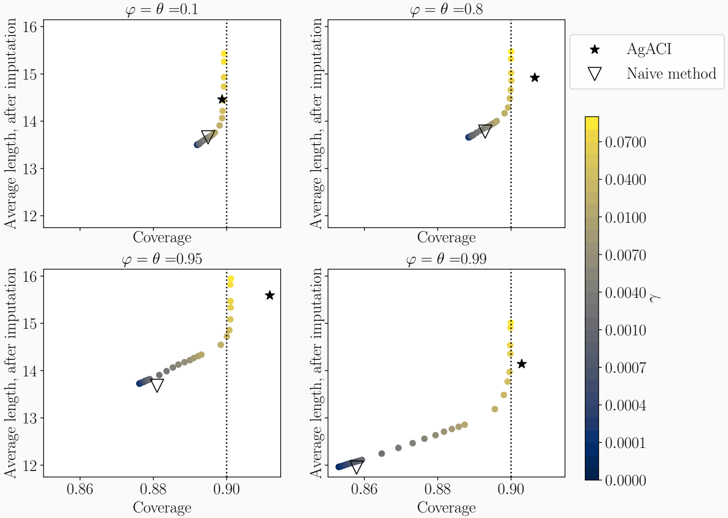
Naive method (▽): smallest among valid ones in the past

Empirical evaluation of ACI sensitivity to γ and adaptive choice



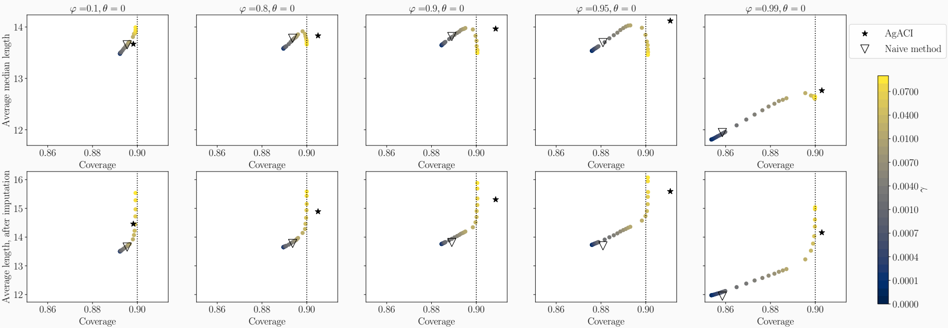
Naive method (∇): smallest among valid ones in the past $\Rightarrow$ accumulates error of the different ACI's versions.

Empirical evaluation of ACI sensitivity to γ and adaptive choice

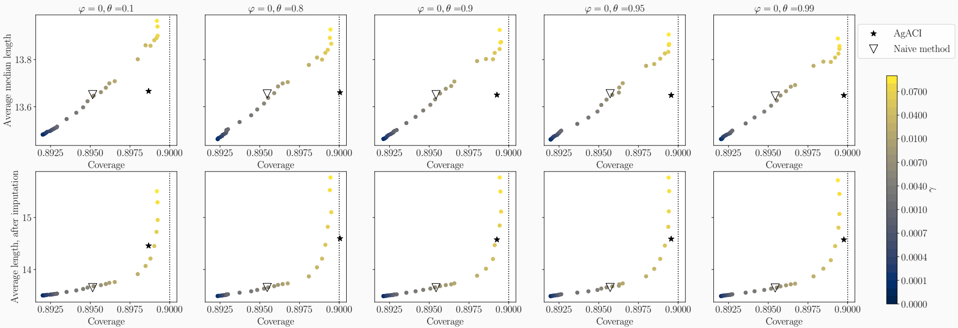


AgACI (★): encouraging preliminary results.

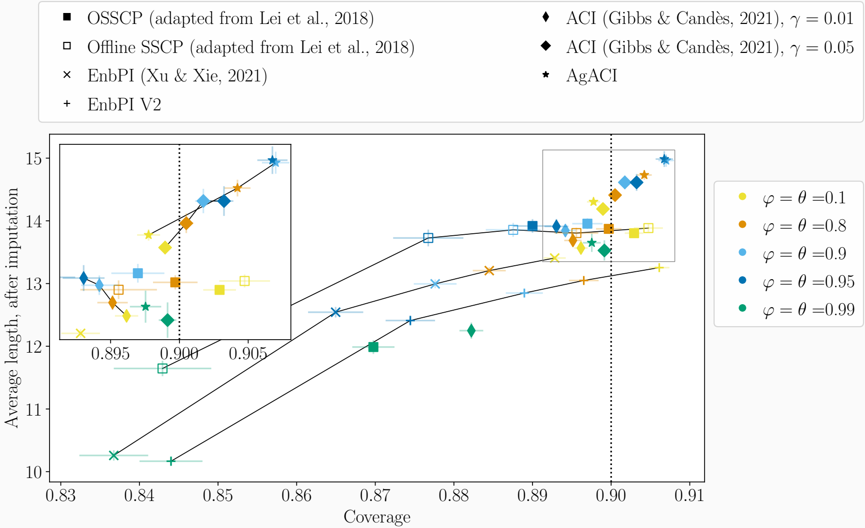
Empirical evaluation of ACI sensitivity to γ and adaptive choice, AR(1)



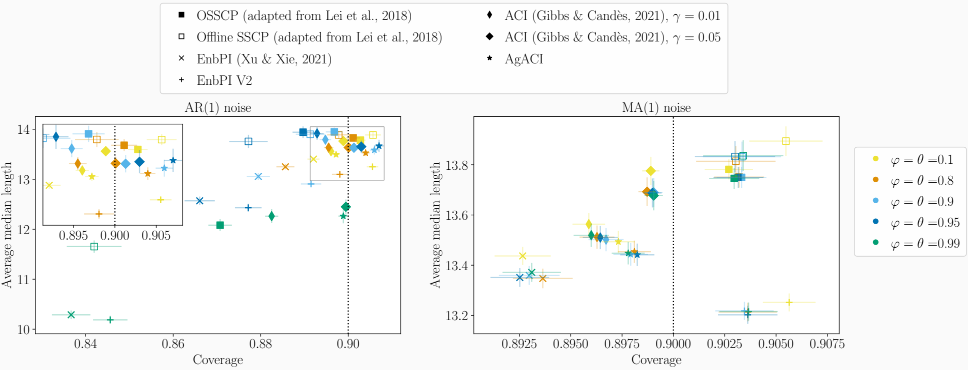
Empirical evaluation of ACI sensitivity to γ and adaptive choice, MA(1)



Results: impact of the temporal dependence, ARMA(1), variance 10, average length after imputation



Results: impact of the temporal dependence, AR(1) and MA(1), variance 10



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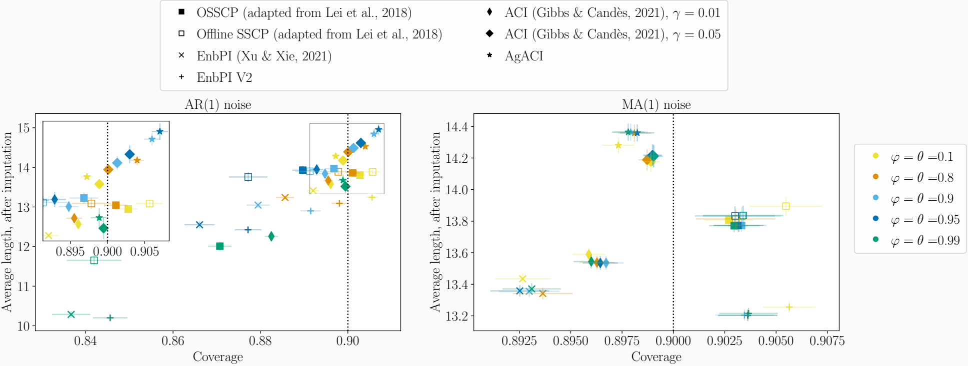

Additional results on the French electricity spot prices

Forecasting French electricity Spot prices with confidence: results

- Target coverage: 90%
- Empirical coverage: 91.68%
- Median length: 22.76€/MWh

Performance on predicted French electricity Spot price: visualisation of a day

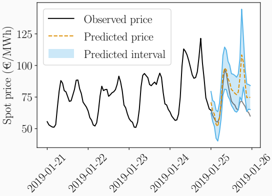

Figure 18: OSSCP

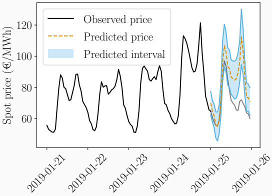

Figure 19: EnbPI V2

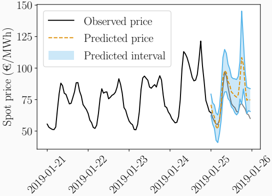

Figure 20: ACI with $\gamma = 0.01$

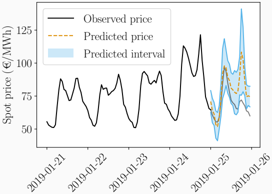

Figure 21: ACI with $\gamma = 0.05$