A new dissimilarity for extreme rainfall clustering, non-parametric and coupling bivariate extreme value theory and marginals

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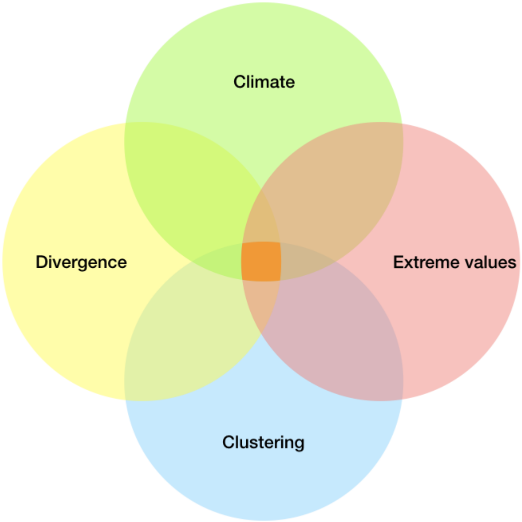
Climate
Divergence
Extreme values
Clustering

Context

Dataset: daily France rainfall in mm ($1\text{mm} = 1\text{L}/\text{m}^2$), from 1976 to 2015, at 174 Meteo-France stations.

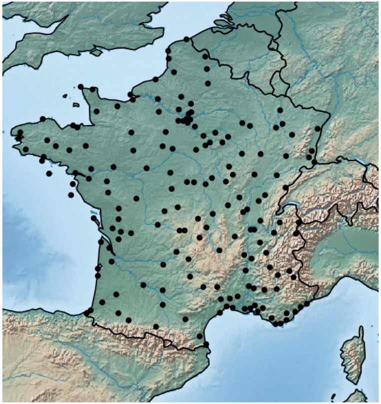

Figure: Weather stations

Objectives

Aim:

- Construct coherent groups of locations according to the nature of the extreme's rainfall/wind/temperature there (e.g. spatial group)

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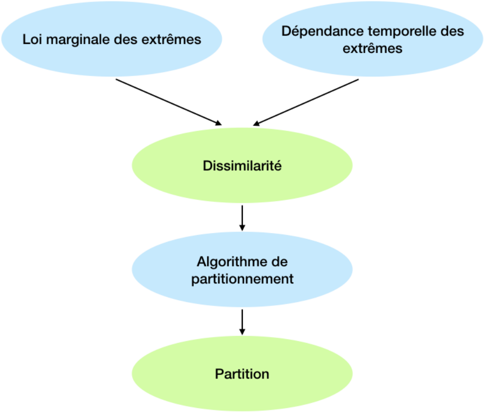
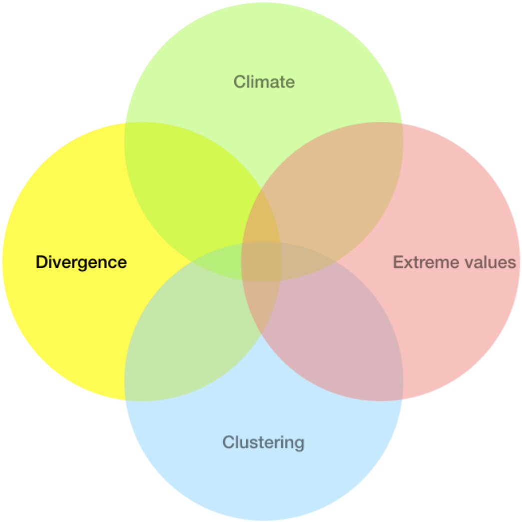
Constraints:

- Non-parametric approach (no fit, few assumptions)
- Good scaling to large datasets

Theoretical tools


Loi marginale des extrêmes
Dépendance temporelle des extrêmes
Dissimilarité
Algorithme de partitionnement
Partition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Divergence

Mathematical tool to quantify the similarity between two probability distributions.

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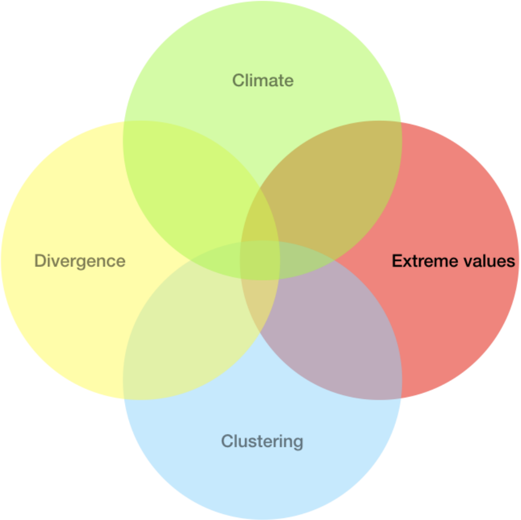
Mathematical tool to quantify the similarity between two probability distributions.

Kullback-Leibler directed divergence is asymmetric. Symmetrized quantity:

Definition (Kullback-Leibler divergence)

Let f and g be two probability density functions. Then:

$$\begin{aligned} D(f,g) &= J(f;g) + J(g;f) \\ &= \mathbb{E}_f\left[\log\left(\frac{f(X)}{g(X)}\right)\right] + \mathbb{E}_g\left[\log\left(\frac{g(Y)}{f(Y)}\right)\right] \end{aligned}$$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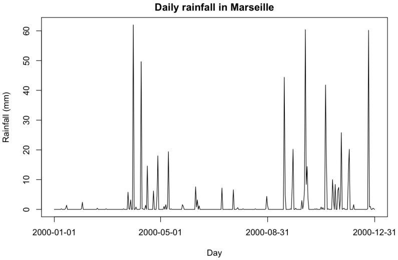
Extreme

Excesses over a threshold.

Variable of interest: $X_u = [X|X > u]$, for some well-chosen and high threshold u.



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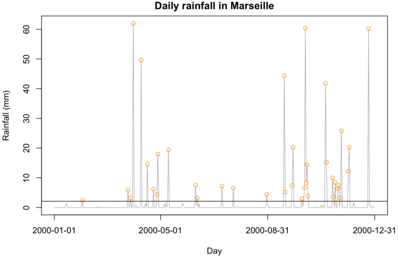

Extreme

Excesses over a threshold.

Variable of interest: $X_u = [X|X > u]$, for some well-chosen and high threshold u.

Let X r.v. with density f (and tail $\overline{F}$). Then, for any threshold u, has the following characteristics:

- density $f_u(x) = \frac{f(x)}{\overline{F}(u)} \mathbb{1}_{\{x>u\}}$
- tail $\overline{F}_u(x) = \frac{\overline{F}(x)}{\overline{F}(u)} \mathbb{1}_{\{x>u\}}$

Kullback-Leibler divergence tailored for large excesses

KL adapted for univariate extremes, in terms of excesses, in [Naveau et al., 2014]. We can consider the quantity:

$$D(f_u, g_u) = J(f_u; g_u) + J(g_u; f_u)$$

with $J(f_u; g_u) = \mathbb{E}_{f_u}\left[\log\left(\frac{f_u(X_u)}{g_u(X_u)}\right)\right]$.

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Proposition (from [Naveau et al., 2014])

$D(f_u, g_u)$ is equivalent (under assumptions), as $u \to \tau$, to:

$$K(f_u, g_u) = -L(f_u; g_u) - L(g_u; f_u)$$

with: $L(f_u; g_u) = \mathbb{E}_f\left[\log\left(\frac{\overline{G}(X)}{\overline{G}(u)}\right)|X > u\right] - 1.$

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→ easy construction of a plug-in estimator

Bivariate extreme dependence

Tail dependency

Take into account the tail dependence.

Definition (residual tail dependence coefficient)

$$\frac{\mathbb{P}\{A\}\mathbb{P}\{B\}}{\mathbb{P}\{A \cap B\}}$$

Tail dependency

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Definition (residual tail dependence coefficient)

Let X, Y, two r.v. of c.d.f F and G respectively.

$$\frac{\mathbb{P}\{X > F^{-1}(q)\}\mathbb{P}\{Y > G^{-1}(q)\}}{\mathbb{P}\{X > F^{-1}(q), Y > G^{-1}(q)\}}$$

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$$\bar{\chi} = \lim_{q \to 1} \frac{\log\left(\mathbb{P}\{X > F^{-1}(q)\}\mathbb{P}\{Y > G^{-1}(q)\}\right)}{\log\left(\mathbb{P}\{X > F^{-1}(q), Y > G^{-1}(q)\}\right)} - 1$$

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$\bar{\chi} \in [-1, 1]$

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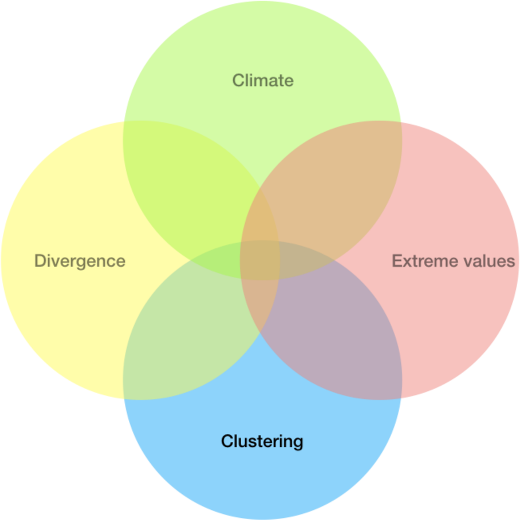
- $\lambda = 1 \Leftrightarrow$ marginal law ;
- $\lambda = 0 \Leftrightarrow$ dependence structure.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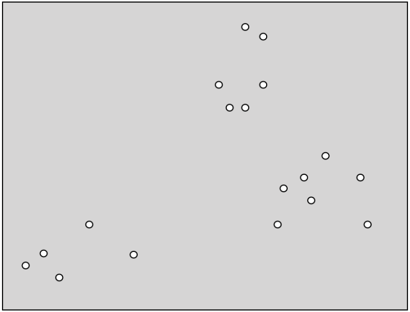
Clustering

We consider a simple and classic algorithm: `Partitioning Around Medoids (PAM)`, also called `k-medoids`, proposed by [Kaufman and Rousseeuw, 1987].



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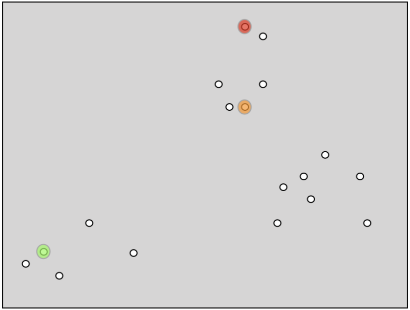

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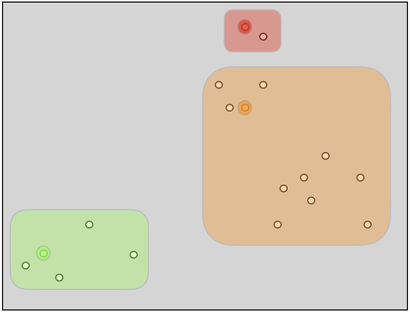

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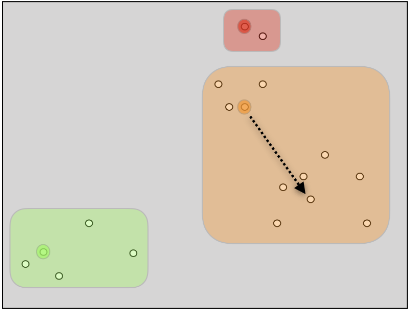

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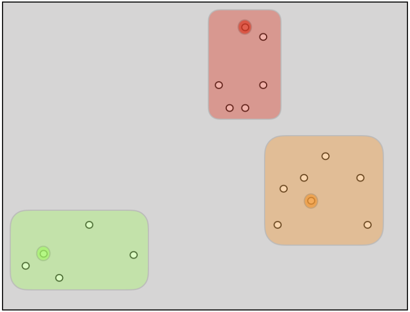

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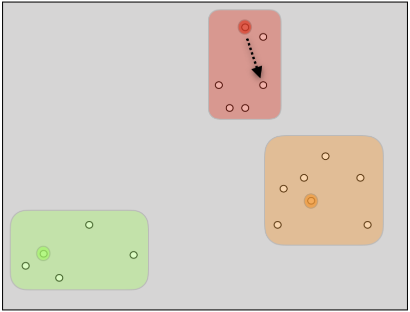

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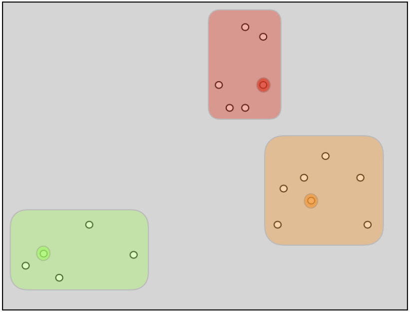

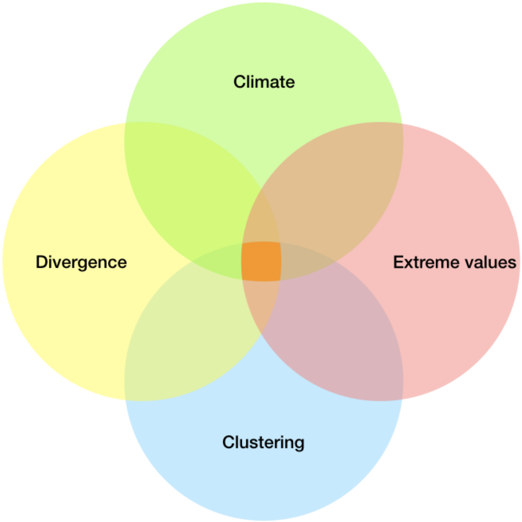
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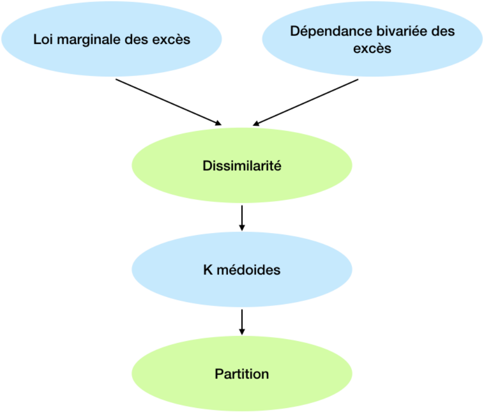
Algorithm


Loi marginale des excès
Dépendance bivariée des excès
Dissimilarité
K médoides
Partition

Back to rainfall

$q = 0.9, \lambda = 0.5$

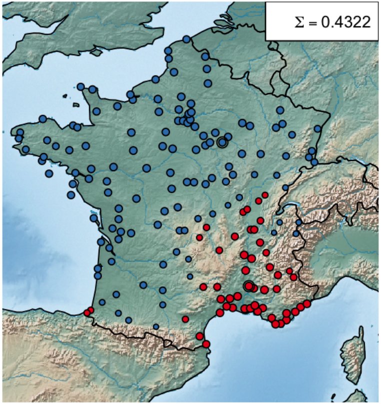


Figure: 2 clusters

$q = 0.9, \lambda = 0.5$

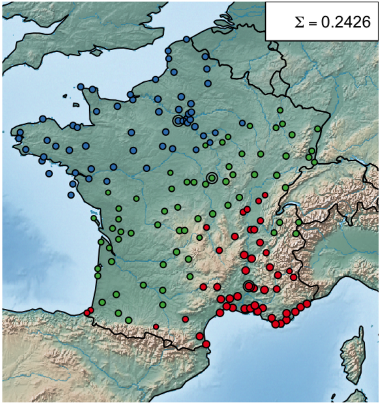


Figure: 3 clusters

$q = 0.9, \lambda = 0.5$

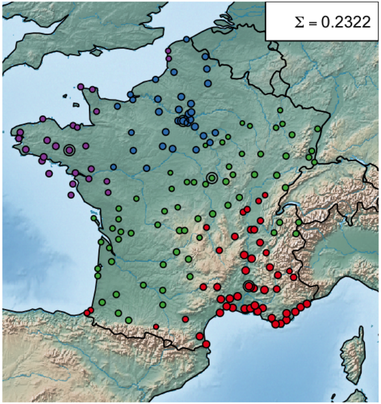


Figure: 4 clusters

Changing points: 3 clusters, $q = 0.9$

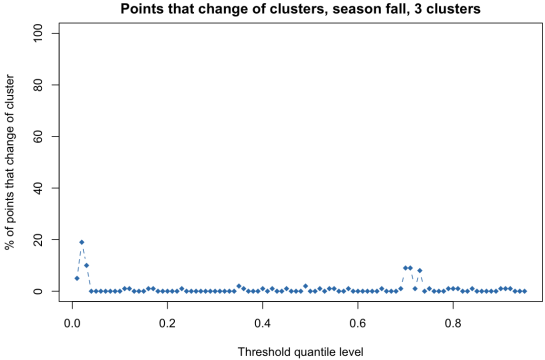


Figure: $\lambda = 0.5$

Constant points: $\lambda = 0.5$, $q_{\min} = 0.74$

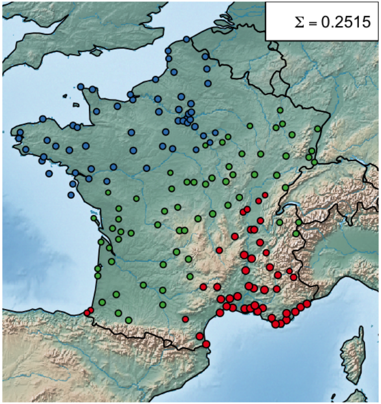


Figure: 3 clusters

Conclusion

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- Climatologically coherent clusters
- Tool to help choosing the threshold level
- Implementation in a R package

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Perspectives:

- Analyze whether the clusters are going to evolve in the climate change framework, using results of simulations
- Choice and cost of the clustering algorithm
- Statistical estimator study

Conclusion

Thanks for your attention!

Bernard, E., Naveau, P., Vrac, M., and Mestre, O. (2013).
Clustering of Maxima: Spatial Dependencies among Heavy Rainfall in France.
Journal of Climate, 26(20):7929–7937.

Kaufman, L. and Rousseeuw, P. J. (1987).
Clustering by means of medoids.

Naveau, P., Guillou, A., and Rietsch, T. (2014).
A non-parametric entropy-based approach to detect changes in climate extremes.
Journal of the Royal Statistical Society: Series B (Statistical Methodology), 76(5):861–884.

Rousseeuw, P. J. (1987).
Silhouettes: A graphical aid to the interpretation and validation of cluster analysis.
Journal of Computational and Applied Mathematics, 20:53 – 65.

Tail dependency, gaussian case

Example (Bivariate Gaussian distribution)

Let $X = (X_1, X_2)$ bivariate gaussian distributed random variable, of correlation parameter ρ.
Then $\bar{\chi}(X_1, X_2) = \rho$

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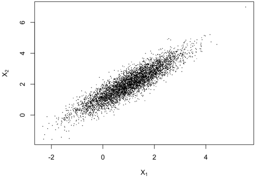
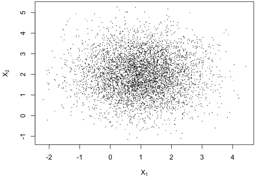


Figure: $\rho = 0.9$



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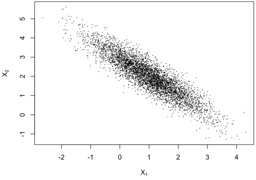
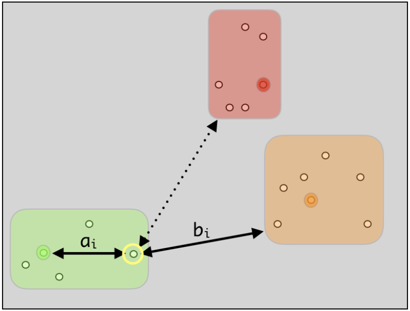


Figure: $\rho = -0.9$

Clustering: evaluation



Clustering: evaluation

Definition (Silhouette coefficient, [Rousseeuw, 1987])

Let's consider a point i.

- a_i the average dissimilarity of point i with the other points of its cluster;
- b_i the minimal average dissimilarity of point i to any other cluster.

We can now define the silhouette coefficient of i by:

$$s_i = \begin{cases} 1 - \frac{a_i}{b_i} & \text{if } a_i < b_i \text{ and } |C_i| > 1 \\ 0 & \text{if } a_i = b_i \text{ or } |C_i| = 1 \\ \frac{b_i}{a_i} - 1 & \text{if } a_i > b_i \text{ and } |C_i| > 1 \end{cases}$$

Changing points: method

Algorithm 1 Detect changing points between C_1 and C_2

Require: 2 clusterings of n points, C_1 and C_2

Initialize empty list: `moving_points`
Build A_1 and A_2, adjacency matrices of C_1 and C_2
$D = A_1 - A_2$
$S_i = \sum_{j=1}^{n} \mathbb{1}_{D_{i,j} \in \{-1,1\}}$
while $S \neq 0 \in \mathbb{R}^n$ **do**
    $u = \underset{i \in [\![1,n]\!]}{\operatorname{argmax}} \{S_i\}$
    Append u to `moving_points`
    $D_{u,\cdot} = 0$ and $D_{\cdot,u} = 0$
    Recompute S
end while
return `moving_points`

$q = 0.9, \lambda = 1$

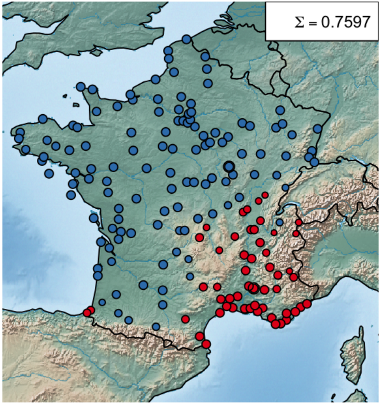


Figure: 2 clusters

$q = 0.9, \lambda = 1$

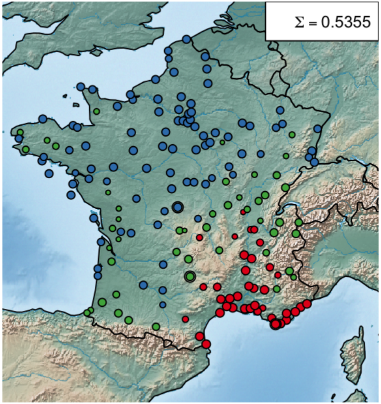


Figure: 3 clusters

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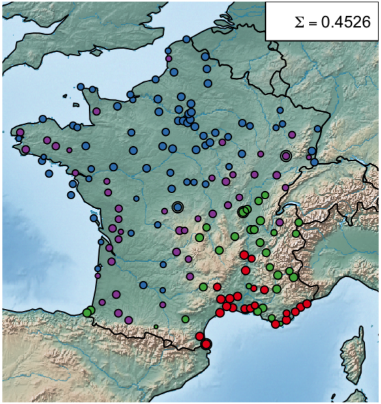

Figure: 4 clusters

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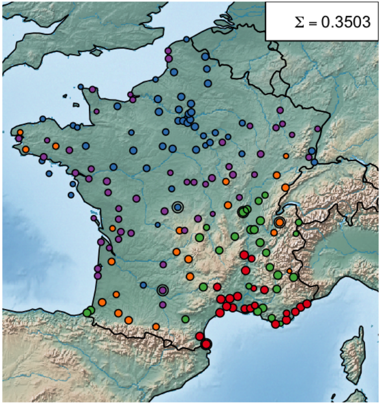

Figure: 5 clusters

q = 0.9, $\lambda = 1$

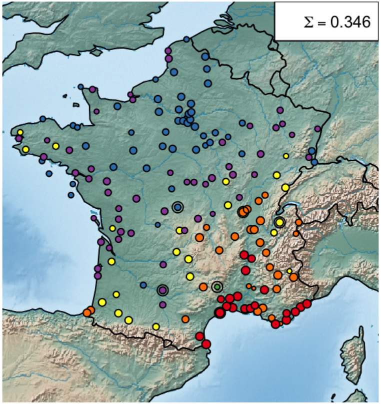

Figure: 6 clusters

$q = 0.9, \lambda = 1$

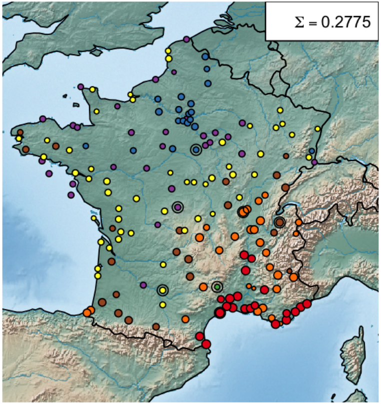

Figure: 7 clusters

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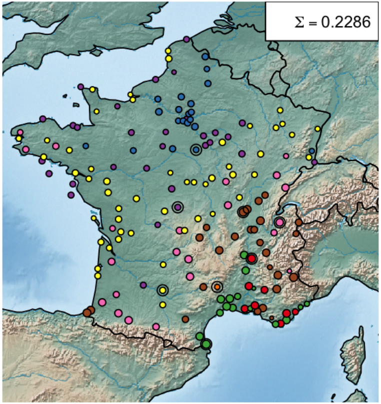

Figure: 8 clusters

$q = 0.9$, $\lambda = 1$

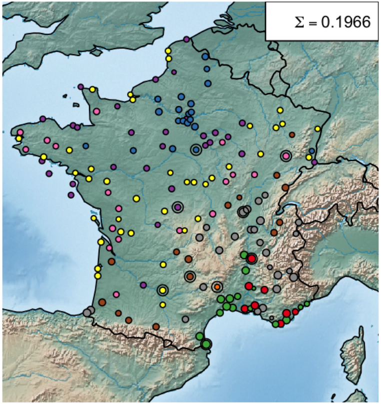

Figure: 9 clusters

$q = 0.9, \lambda = 0.75$

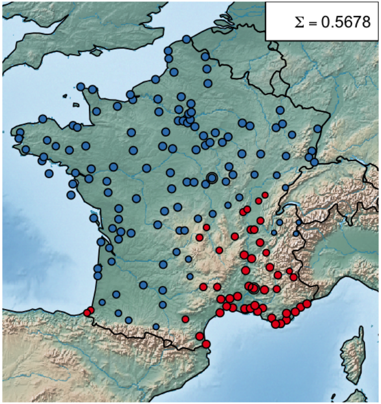

Figure: 2 clusters

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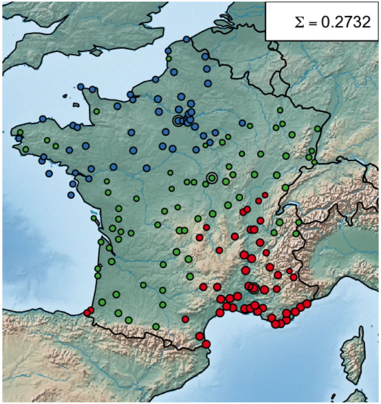

Figure: 3 clusters

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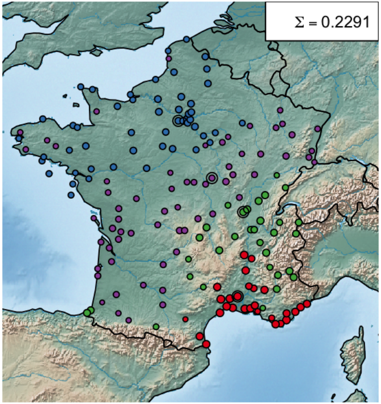

Figure: 4 clusters

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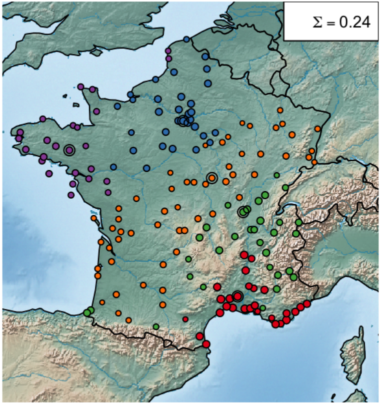

Figure: 5 clusters

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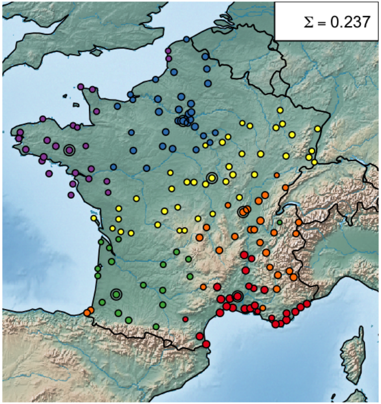

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$q = 0.9, \lambda = 0.75$

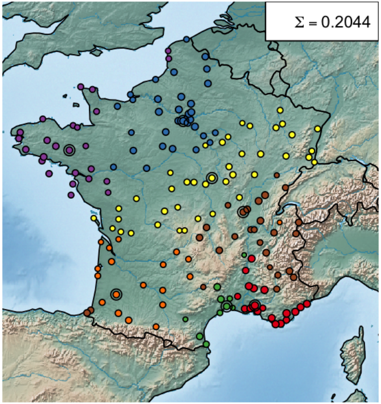


Figure: 7 clusters

$q = 0.9, \lambda = 0.75$

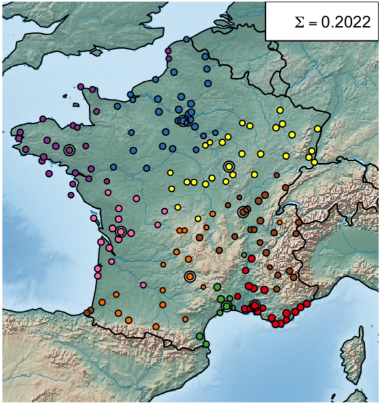


Figure: 8 clusters

$q = 0.9, \lambda = 0.75$

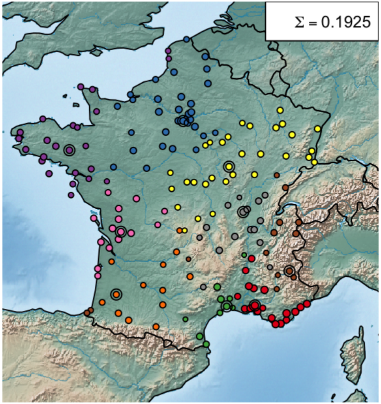


Figure: 9 clusters

q = 0.9, λ = 0.5

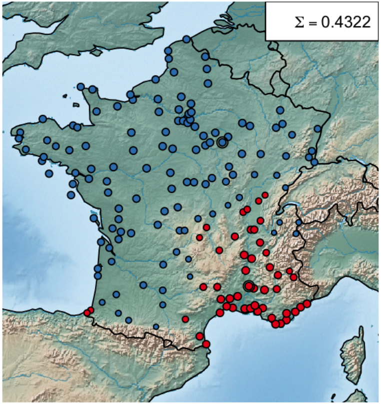


Figure: 2 clusters

q = 0.9, $\lambda = 0.5$

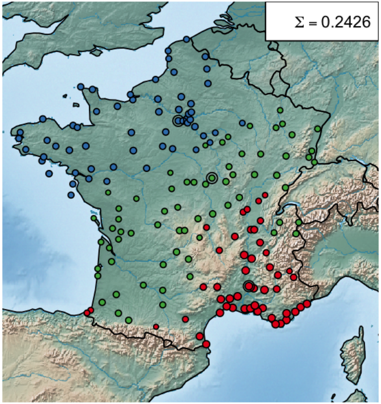


Figure: 3 clusters

$q = 0.9, \lambda = 0.5$

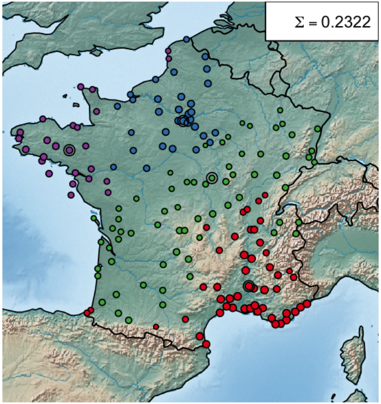


Figure: 4 clusters

$q = 0.9, \lambda = 0.5$

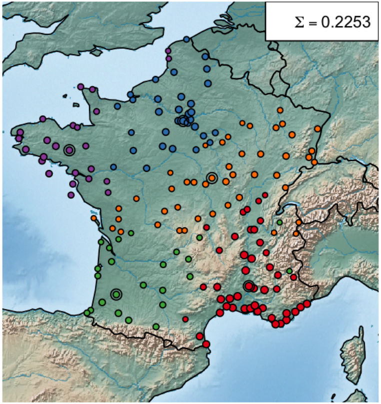


Figure: 5 clusters

$q = 0.9, \lambda = 0.5$

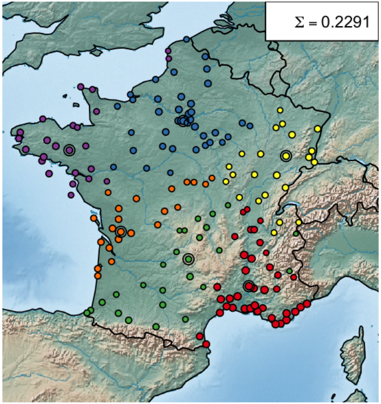


Figure: 6 clusters

$q = 0.9, \lambda = 0.5$

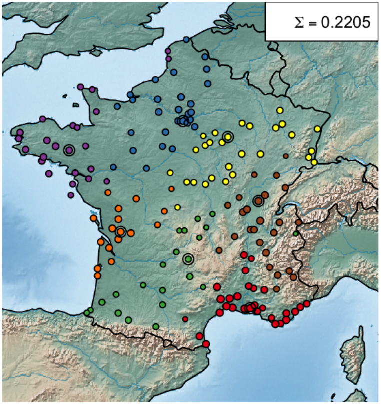


Figure: 7 clusters

$q = 0.9, \lambda = 0.5$

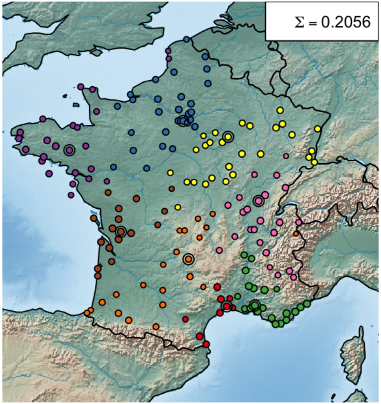


Figure: 8 clusters

$q = 0.9$, $\lambda = 0.5$

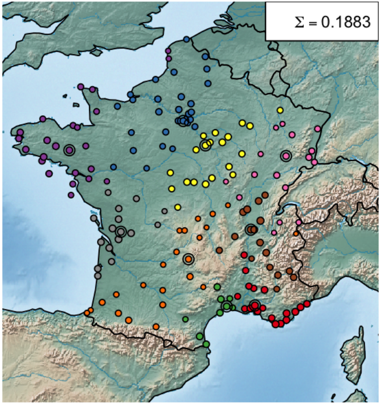


Figure: 9 clusters

q = 0.9, $\lambda = 0.25$

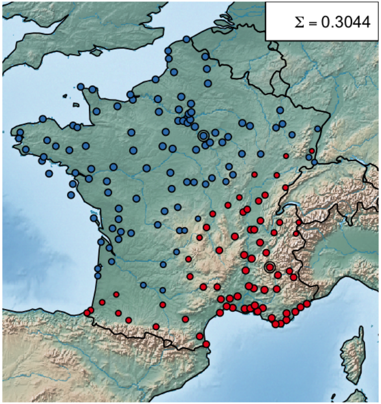


Figure: 2 clusters

$q = 0.9, \lambda = 0.25$

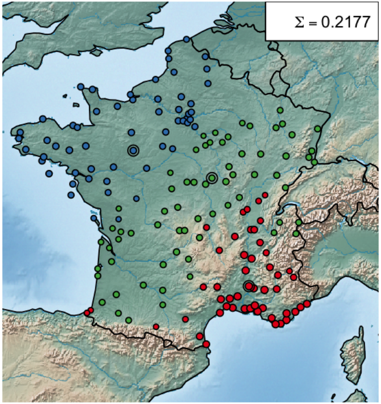


Figure: 3 clusters

$q = 0.9, \lambda = 0.25$

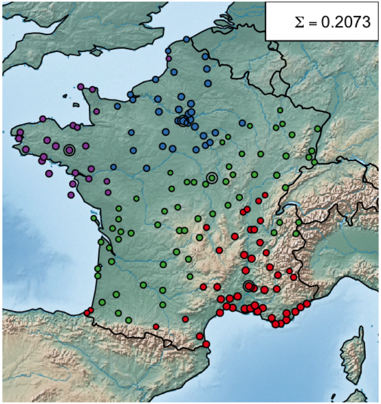


Figure: 4 clusters

$q = 0.9, \lambda = 0.25$

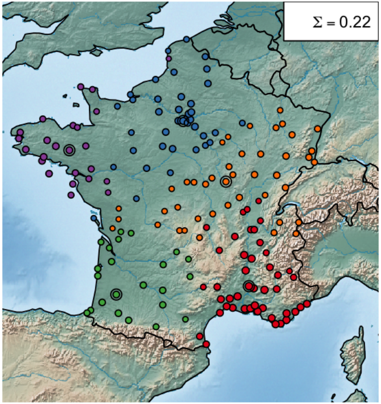


Figure: 5 clusters

$q = 0.9, \lambda = 0.25$

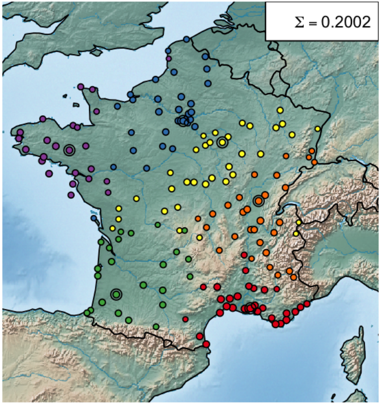


Figure: 6 clusters

$q = 0.9, \lambda = 0.25$

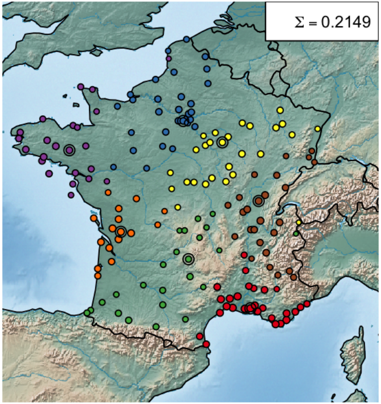


Figure: 7 clusters

$q = 0.9, \lambda = 0.25$

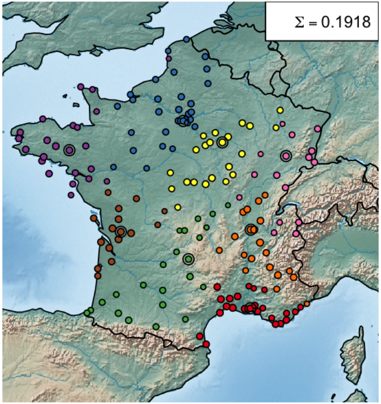


Figure: 8 clusters

$q = 0.9, \lambda = 0.25$

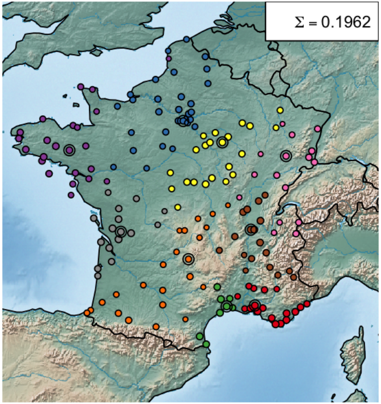


Figure: 9 clusters

$q = 0.9, \lambda = 0$

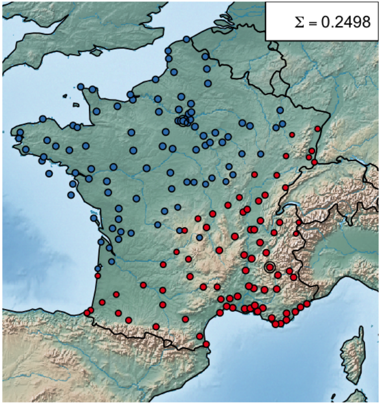


Figure: 2 clusters

$q = 0.9, \lambda = 0$

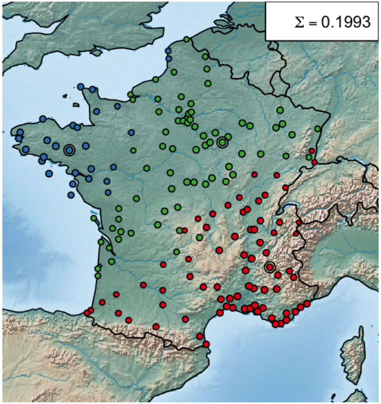


Figure: 3 clusters

$q = 0.9, \lambda = 0$

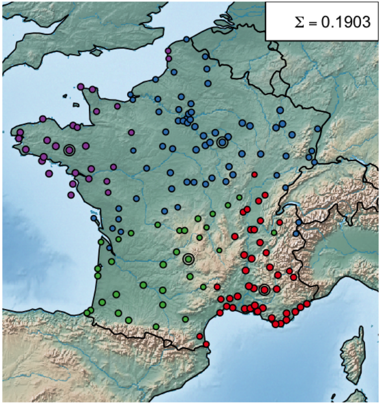


Figure: 4 clusters

$q = 0.9, \lambda = 0$

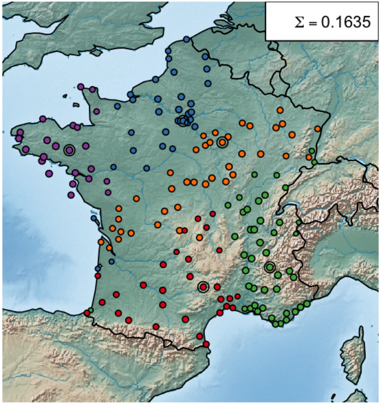


Figure: 5 clusters

$q = 0.9, \lambda = 0$

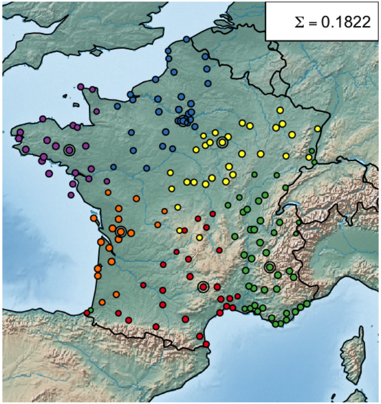


Figure: 6 clusters

$q = 0.9, \lambda = 0$

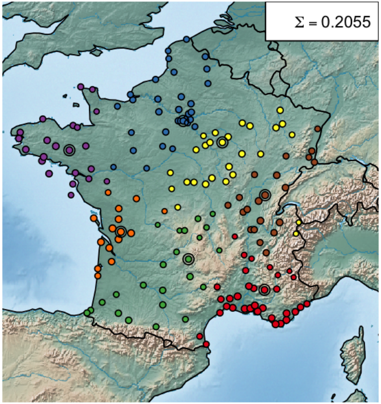


Figure: 7 clusters

$q = 0.9, \lambda = 0$

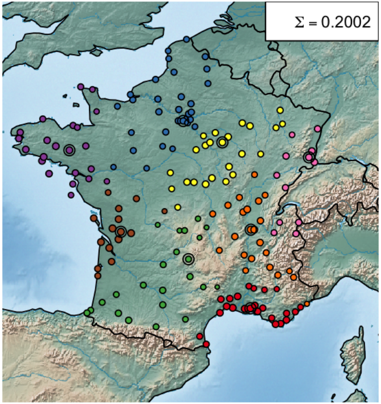


Figure: 8 clusters

$q = 0.9$, $\lambda = 0$

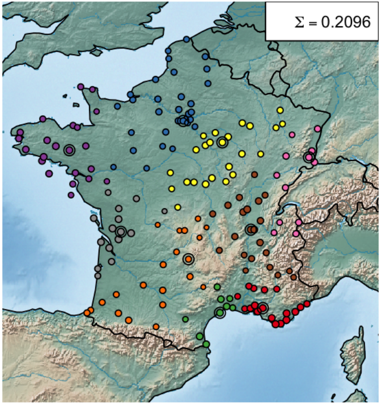


Figure: 9 clusters

Silhouette with number of clusters

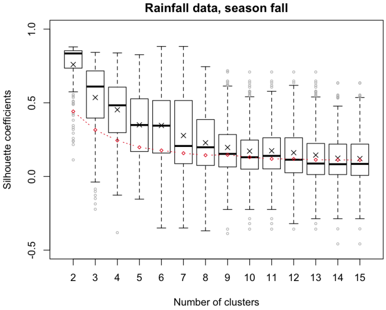


Figure: $\lambda = 1$

Silhouette with number of clusters

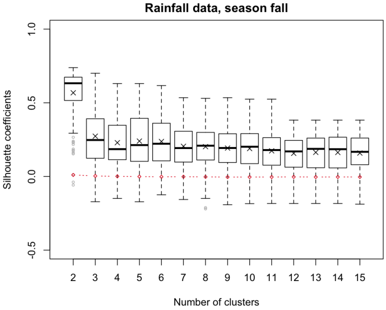


Figure: $\lambda = 0.75$

Silhouette with number of clusters

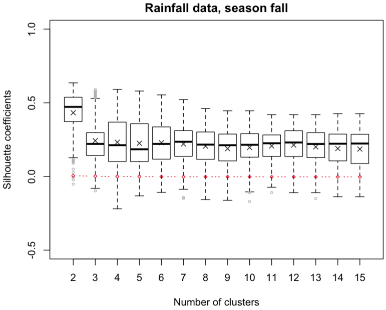


Figure: $\lambda = 0.5$

Silhouette with number of clusters

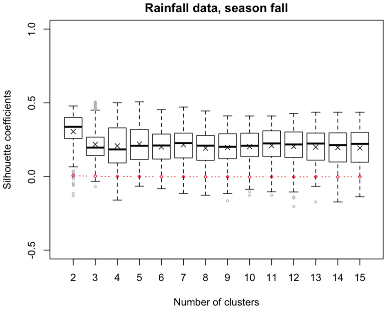


Figure: $\lambda = 0.25$

Silhouette with number of clusters

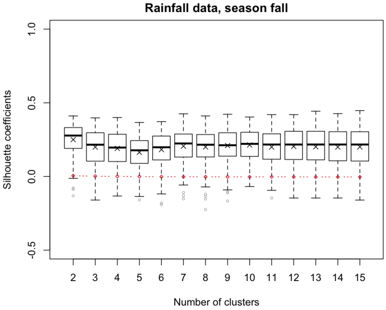
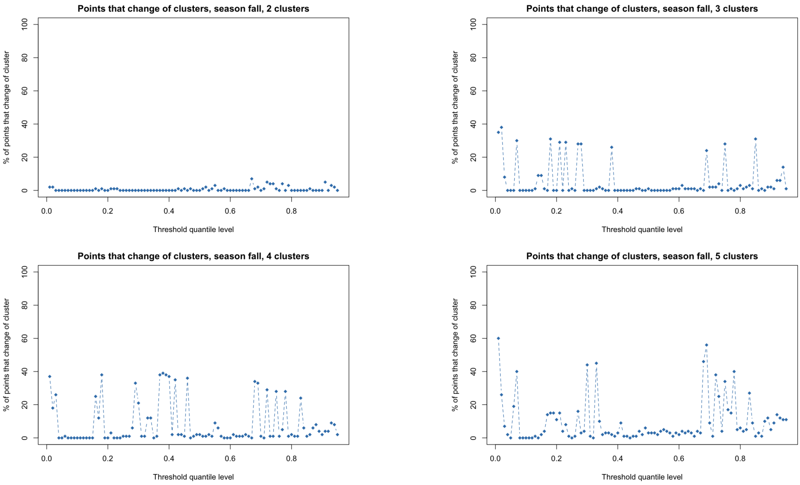
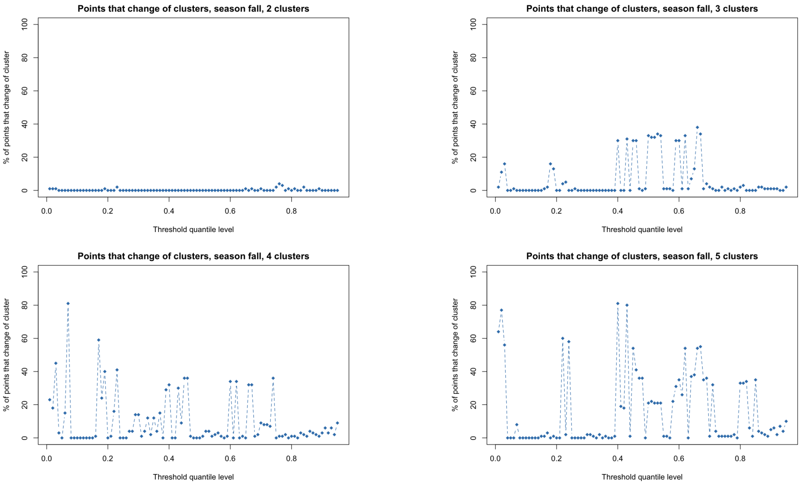


Figure: $\lambda = 0$

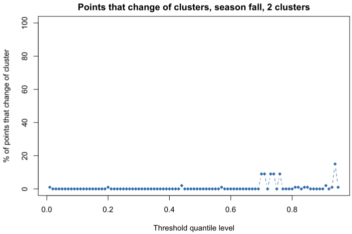
Changing points, $\lambda = 1$

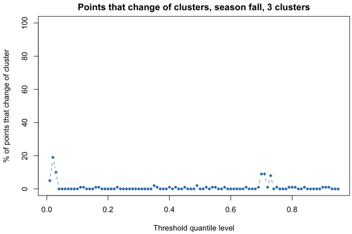


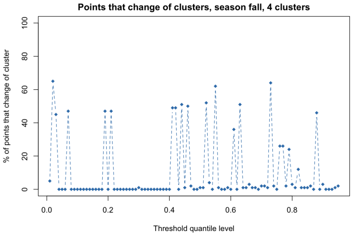
Changing points, $\lambda = 0.75$

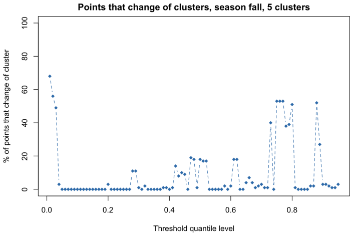


Changing points, $\lambda = 0.5$

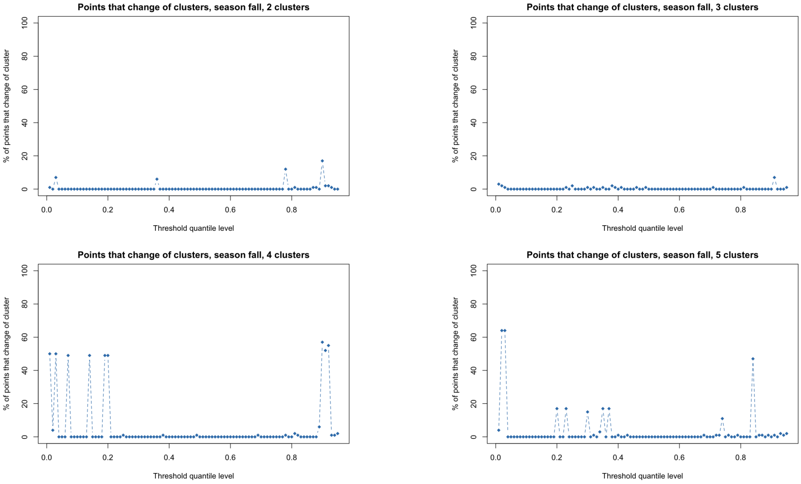








Changing points, $\lambda = 0.25$



No changing, 2 clusters

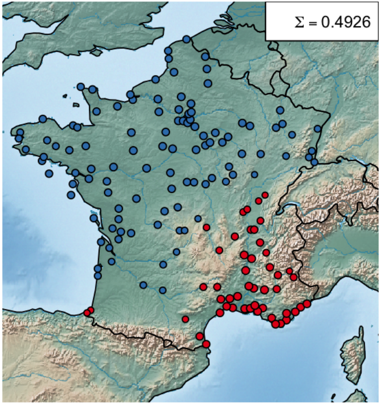


Figure: $q_{\min} = 0.77$

No changing, 3 clusters

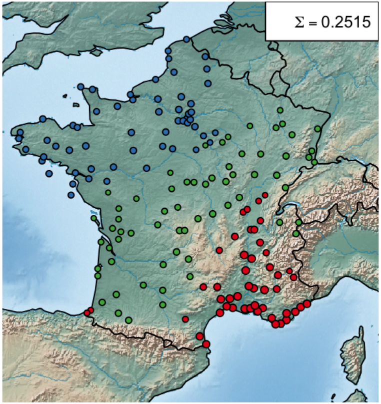


Figure: $q_{\text{min}} = 0.74$

No changing, 4 clusters

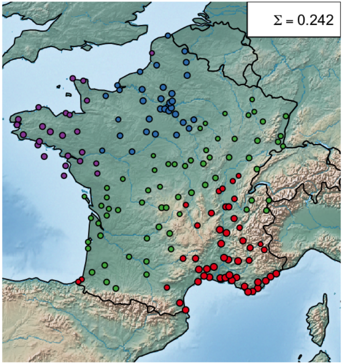


Figure: $q_{\text{min}} = 0.89$

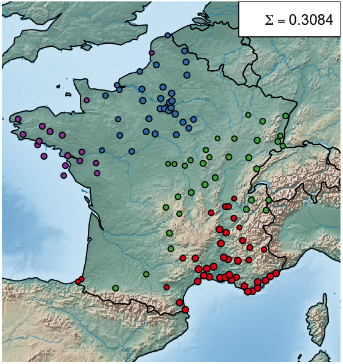


Figure: $q_{\text{min}} = 0.83$

No changing, 5 clusters

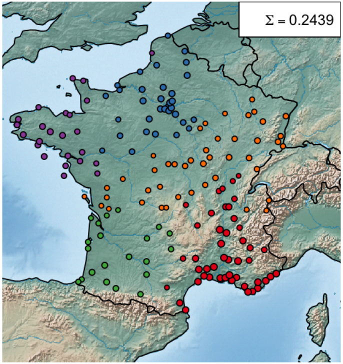


Figure: $q_{\text{min}} = 0.9$

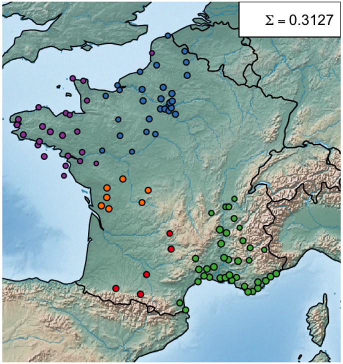


Figure: $q_{\text{min}} = 0.81$

[Bernard et al., 2013]

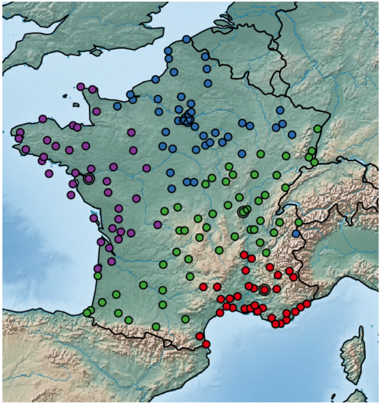


Figure: 4 clusters